Bi-Hamiltonian structure of spin Sutherland models from Poisson reduction

László Fehér, University of Szeged and Wigner RCP, Budapest

Calogero–Moser–Sutherland type integrable many-body models appear in several fields of physics, and still attract lot of attention due to their rich mathematical structure. A prime example is the trigonometric Sutherland model governed by the Hamiltonian

$$H_{\mathrm{Suth}}(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{8}\sum_{k\neq l} \frac{x}{\sin^2 \frac{q_k - q_l}{2}}, \quad \text{with real coupling constant} \quad x > 0.$$

Due to Olshanetsky–Perelomov (1976) and Kazhdan–Kostant–Sternberg (1978), this model can be interpreted as a symplectic reduction of the 'free particle' moving on the unitary group U(n). The reduction uses the conjugation action of U(n) on T^*U(n), and relies on fixing the relevant moment map to a very specific value.

Allowing arbitrary moment map values, the reduction of T^*U(n) leads to the trigonometric spin Sutherland model having the 'main Hamiltonian'

$$H_{\mathrm{spin-Suth}}(q,p,\phi) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \frac{1}{8}\sum_{k\neq l} \frac{\phi_{kl}\phi_{lk}}{\sin^2 \frac{q_k - q_l}{2}}, \quad \phi_{kl}\phi_{lk} = |\phi_{kl}|^2,$$

where the 'collective spin variable' $\phi \in \mathfrak{u}(n)^* \simeq \sqrt{-1}\mathfrak{u}(n)$ has zero diagonal part.

The holomorphic spin Sutherland model descends by Poisson reduction from the holomorphic cotangent bundle $T^*\mathrm{GL}(n,\mathbb{C})$, and its trigonometric and hyperbolic real forms descend from the real cotangent bundles $T^*\mathrm{U}(n)$ and $T^*P(n)$, respectively, where $P(n)$ is the symmetric space $\mathrm{GL}(n,\mathbb{C})_\mathbb{R}/\mathrm{U}(n)$ with $\mathrm{GL}(n,\mathbb{C})_\mathbb{R}$ denoting the realification of $\mathrm{GL}(n,\mathbb{C})$. My basic observation is that the cotangent bundles

$$T^*\mathrm{GL}(n,\mathbb{C}), \quad T^*\mathrm{U}(n), \quad T^*\mathrm{GL}(n,\mathbb{C})_\mathbb{R}$$

are bi-Hamiltonian manifolds, and the 'free Hamiltonians' of these phase spaces form bi-Hamiltonian hierarchies. By taking Poisson quotient, bi-Hamiltonian spin Sutherland models result.

Application of the same idea to $T^*\mathrm{GL}(n,\mathbb{R})$ leads to the bi-Hamiltonian structure on the associative algebra $\mathfrak{gl}(n,\mathbb{R})$ that underlies the linear and quadratic Poisson structures of the open Toda lattice.

What is a bi-Hamiltonian system?

We have a classical phase space, whose space of functions carries two Poisson brackets $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ such that the time evolution of any function F can be written alternatively as

$$\dot{F} = \{F, H_1\}_2 = \{F, H_2\}_1 \quad \text{with Hamiltonians } H_1 \text{ and } H_2.$$

The two Poisson brackets are supposed to be compatible, which means that any linear combination

$$\lambda_1\{\ ,\ \}_1 + \lambda_2\{\ ,\ \}_2$$

satisfies the Jacobi identity (λ_1 and λ_2 are arbitrary constants).

Many classical integrable systems are bi-Hamiltonian. A basic fact is that if the recursion (so-called Magri–Lenard scheme)

$$\{\cdot\,, H_m\}_2 = \{\cdot\,, H_{m+1}\}_1 \quad \text{say for all} \quad m \in \mathbb{N}$$

holds, then $\{H_m, H_n\}_1 = \{H_m, H_n\}_2 = 0$. Hence we have a set of commuting Hamiltonians. Under favourable circumstances, they are part of an integrable Hamiltonian system.

A well-known lemma about getting compatible Poisson brackets

Lemma 0. *Let* $(\mathfrak{A}, \{\ ,\ \})$ *be a Poisson algebra and* $\mathcal{D}$ *a derivation of the underlying commutative algebra* $\mathfrak{A}$ *(of functions on the phase space). Suppose that the bracket*

$$\{f, h\}^{\mathcal{D}} := \mathcal{D}[\{f, h\}] - \{\mathcal{D}[f], h\} - \{f, \mathcal{D}[h]\}$$

satisfies the Jacobi identity. Then the formula

$$\{f, h\}_{\lambda_1,\lambda_2} = \lambda_1\{f, h\} + \lambda_2\{f, h\}^{\mathcal{D}}$$

defines a Poisson bracket, for any constant parameters λ_1 *and* λ_2*.*

Note: For any derivation $\mathcal{D}$, the bracket $\{\ ,\ \}_{\lambda_1,\lambda_2}$ is automatically anti-symmetric and verifies the Leibniz property. It is a simple exercise to verify the Jacobi identity by direct calculation.

Plan of the talk

- The holomorphic cotangent bundle of $\mathrm{GL}(n, \mathbb{C})$ and its reduction

- Bi-hamiltonian structure on $\mathfrak{gl}(n, \mathbb{R})$ from reduction of $T^*\mathrm{GL}(n, \mathbb{R})$

- Spin Sutherland models coupled to two spins from $T^*\mathrm{GL}(n, \mathbb{C})_{\mathbb{R}}$

- Concluding remarks

On the holomorphic cotangent bundle of $\mathrm{GL}(n,\mathbb{C})$

Denote $G := \mathrm{GL}(n,\mathbb{C})$ and equip $\mathcal{G} := \mathfrak{gl}(n,\mathbb{C})$ with the trace form $\langle X, Y\rangle := \mathrm{tr}(XY)$.

$$\text{Consider} \quad T^*G \simeq G \times \mathcal{G} = \{(g,L) \mid g \in G,\ L \in \mathcal{G}\} =: \mathfrak{M},$$

and let $\mathrm{Hol}(\mathfrak{M})$ be the commutative algebra of holomorphic functions on $\mathfrak{M}$. For any $F \in \mathrm{Hol}(\mathfrak{M})$, define the $\mathcal{G}$-valued derivatives $\nabla_1 F$, $\nabla_1' F$ and $d_2 F$ by

$$\langle \nabla_1 F(g,L), X\rangle = \frac{d}{dz}\bigg|_{z=0} F(e^{zX}g, L), \quad \langle \nabla_1' F(g,L), X\rangle = \frac{d}{dz}\bigg|_{z=0} F(ge^{zX}, L), \quad \forall X \in \mathcal{G},$$

and $\langle d_2F(g,L), X\rangle = \frac{d}{dz}\big|_{z=0} F(g, L+zX)$. Introduce also

$$\nabla_2 F(g,L) := L d_2 F(g,L), \qquad \nabla_2' F(g,L) := (d_2F(g,L))L.$$

By the triangular decomposition, $\mathcal{G} = \mathcal{G}_> + \mathcal{G}_0 + \mathcal{G}_<$, write $\forall X \in \mathcal{G}$ as $X = X_> + X_0 + X_<$. Define the classical r-matrix $r \in \mathrm{End}(\mathcal{G})$ by $r(X) := \frac{1}{2}(X_> - X_<)$, and put $r_\pm := r \pm \frac{1}{2}\mathrm{id}$.

Theorem 1. For functions $F, H \in \mathrm{Hol}(\mathfrak{M})$, the following formulae define two Poisson brackets:

$$\{F,H\}_1(g,L) = \langle \nabla_1 F, d_2 H\rangle - \langle \nabla_1 H, d_2 F\rangle + \langle L, [d_2F, d_2H]\rangle, \tag{1}$$

and

$$\begin{aligned}\{F,H\}_2(g,L) \;=\; & \langle r\nabla_1 F, \nabla_1 H\rangle - \langle r\nabla_1' F, \nabla_1' H\rangle \\ & + \langle \nabla_2 F - \nabla_2' F, r_+\nabla_2' H - r_-\nabla_2 H\rangle \\ & + \langle \nabla_1 F, r_+\nabla_2' H - r_-\nabla_2 H\rangle - \langle \nabla_1 H, r_+ \nabla_2' F - r_-\nabla_2 F\rangle, \end{aligned} \tag{2}$$

where the derivatives are evaluated at (g,L), and we put rX for $r(X)$.

Theorem 2. The first Poisson bracket of Theorem 1 is the Lie derivative of the second Poisson bracket along the holomorphic vector field on $\mathfrak{M}$ whose integral curve through the initial value (g, L) is

$$\phi_z(g, L) = (g, L + z\mathbf{1}_n), \quad z \in \mathbb{C},$$

where $\mathbf{1}_n$ is the unit matrix. Consequently, the two Poisson brackets are compatible

Denote by V_H^i $(i = 1, 2)$ the Hamiltonian vector field associated with the holomorphic function H through the respective Poisson bracket $\{\ ,\ \}_i$. For any holomorphic function, we have the derivatives $V_H^i[F] = \{F, H\}_i$. We are interested in the Hamiltonians

$$H_m(g, L) := \frac{1}{m}\mathrm{tr}(L^m), \qquad \forall m \in \mathbb{N}. \tag{3}$$

Proposition 3. The vector fields associated with the functions H_m are bi-Hamiltonian:

$$\{F, H_m\}_2 = \{F, H_{m+1}\}_1, \qquad \forall m \in \mathbb{N}, \quad \forall F \in \mathrm{Hol}(\mathfrak{M}). \tag{4}$$

The derivatives of the matrix elements of $(g, L) \in \mathfrak{M}$ give

$$V^2_{H_m}[g] = V^1_{H_{m+1}}[g] = L^m g, \qquad V^2_{H_m}[L] = V^1_{H_{m+1}}[L] = 0, \quad \forall m \in \mathbb{N}, \tag{5}$$

and the flow of $V^2_{H_m} = V^1_{H_{m+1}}$ through the initial value $(g(0), L(0))$ is

$$(g(z), L(z)) = (\exp(zL(0)^m)g(0), L(0)). \tag{6}$$

Remark. The first bracket is linear in the matrix element variables and is just the canonical Poisson bracket of the cotangent bundle. The second one is quadratic, and is obtained from Semenov-Tian-Shanksy's Heisenberg double $G \times G$ of the standard Poisson–Lie group structure on G by a local change of variables and analytic continuation.

The essence of Hamiltonian symmetry reduction is that one keeps only the 'observables' that are invariant with respect to the pertinent group action. This is applicable if, and only if, the invariant functions form a Poisson subalgebra; which is identified with the Poisson algebra of functions on the quotient space.

We apply this principle to the adjoint action of G on $\mathfrak{M}$, for which $\eta \in G$ acts by the holomorphic diffeomorphism A_η,

$$A_\eta : (g, L) \mapsto (\eta g \eta^{-1}, \eta L \eta^{-1}). \tag{7}$$

Thus we keep only the G-invariant holomorphic functions on $\mathfrak{M}$, whose set is denoted

$$\mathrm{Hol}(\mathfrak{M})^G := \{F \in \mathrm{Hol}(\mathfrak{M}) \mid F(g, L) = F(\eta g \eta^{-1}, \eta L \eta^{-1}),\ \forall (g, L) \in \mathfrak{M},\ \eta \in G\}. \tag{8}$$

Lemma 4. For $F, H \in \mathrm{Hol}(\mathfrak{M})^G$, the second Poisson bracket (2) simplifies to

$$2\{F, H\}_2 = \langle \nabla_1 F, \nabla_2 H + \nabla_2' H\rangle - \langle \nabla_1 H, \nabla_2 F + \nabla_2' F\rangle + \langle \nabla_2 F, \nabla_2' H\rangle - \langle \nabla_2 H, \nabla_2' F\rangle. \tag{9}$$

Therefore, $\mathrm{Hol}(\mathfrak{M})^G$ is closed with respect to both Poisson brackets of Theorem 1.

This follows from (2) using the infinitesimal invariance, $\nabla_1' H = \nabla_1 H + \nabla_2 H - \nabla_2' H$, and similar for F.

We need to fix notations. First, define

$$G_0 := \{Q \mid Q = \mathrm{diag}(Q_1, \dots, Q_n),\ Q_i \in \mathbb{C}^*\} < G, \tag{10}$$

and its regular part G_0^{reg}, where $Q_i \neq Q_j$ for all $i \neq j$. Let $\mathcal{N}$ be normalizer of G_0 in G, for which $\mathcal{N}/G_0 = S_n$, and let $G^{\mathrm{reg}} \subset G$ denote the dense open subset consisting of the conjugacy classes having representatives in G_0^{reg}. Next, define

$$\mathfrak{M}^{\mathrm{reg}} := \{(g, L) \in \mathfrak{M} \mid g \in G^{\mathrm{reg}}\} \quad \text{and} \quad \mathfrak{M}_0^{\mathrm{reg}} := \{(Q, L) \in \mathfrak{M} \mid Q \in G_0^{\mathrm{reg}}\}. \tag{11}$$

We introduce the chain of commutative algebras

$$\mathrm{Hol}(\mathfrak{M})_{\mathrm{red}} \subset \mathrm{Hol}(\mathfrak{M}_0^{\mathrm{reg}})^{\mathcal{N}} \subset \mathrm{Hol}(\mathfrak{M}_0^{\mathrm{reg}})^{G_0}. \tag{12}$$

By definition, $\mathrm{Hol}(\mathfrak{M})_{\mathrm{red}}$ *contains the restrictions of the elements of* $\mathrm{Hol}(\mathfrak{M})^G$ *to* $\mathfrak{M}_0^{\mathrm{reg}}$, and the last two sets contain the respective invariant elements of $\mathrm{Hol}(\mathfrak{M}_0^{\mathrm{reg}})$.

Let $\iota : \mathfrak{M}_0^{\mathrm{reg}} \to \mathfrak{M}$ be the tautological embedding. Then pull-back by ι provides an isomorphism between $\mathrm{Hol}(\mathfrak{M})^G$ and $\mathrm{Hol}(\mathfrak{M})_{\mathrm{red}}$. Similarly, $\iota^* : \mathrm{Hol}(\mathfrak{M}^{\mathrm{reg}})^G \to \mathrm{Hol}(\mathfrak{M}_0^{\mathrm{reg}})^{\mathcal{N}}$ is an isomorphism.

Definition 5. Let $f, h \in \mathrm{Hol}(\mathfrak{M})_{\mathrm{red}}$ be related to $F, H \in \mathrm{Hol}(\mathfrak{M})^G$ by $f = F \circ \iota$ and $h = H \circ \iota$. Then we can define $\{f, h\}_i^{\mathrm{red}} \in \mathrm{Hol}(\mathfrak{M})_{\mathrm{red}}$ by the relation

$$\{f, h\}_i^{\mathrm{red}} := \{F, H\}_i \circ \iota, \qquad i = 1, 2. \tag{13}$$

This gives rise to the *reduced Poisson algebras* $(\mathrm{Hol}(\mathfrak{M})_{\mathrm{red}}, \{\ ,\ \}_i^{\mathrm{red}})$.

Any $f \in \mathrm{Hol}(\mathfrak{M}_0^{\mathrm{reg}})$ has the $\mathcal{G}_0$-valued derivative $\nabla_1 f$ and the $\mathcal{G}$-valued derivative $d_2 f$, defined ($\forall X_0 \in \mathcal{G}_0$, $X \in \mathcal{G}$) by

$$\langle \nabla_1 f(Q,L), X_0 \rangle = \frac{d}{dz}\Big|_{z=0} f(e^{zX_0}Q, L), \quad \langle d_2 f(Q,L), X \rangle = \frac{d}{dz}\Big|_{z=0} f(Q, L+zX). \quad (14)$$

Theorem 6. For $f, h \in \mathrm{Hol}(\mathfrak{M})_{\mathrm{reg}}$, the reduced Poisson brackets defined by (13) can be described explicitly as follows:

$$\{f,h\}_1^{\mathrm{red}}(Q,L) = \langle \nabla_1 f, d_2 h \rangle - \langle \nabla_1 h, d_2 f \rangle + \langle L, [d_2 f, \mathcal{R}(Q) d_2 h] + [\mathcal{R}(Q) d_2 f, d_2 h] \rangle, \quad (15)$$

and

$$\{f,h\}_2^{\mathrm{red}}(Q,L) = \langle \nabla_1 f, \nabla_2 h \rangle - \langle \nabla_1 h, \nabla_2 f \rangle + \langle \nabla_2 f, \mathcal{R}(Q)(\nabla_2 h) \rangle - \langle \nabla_2' f, \mathcal{R}(Q)(\nabla_2' h) \rangle, \quad (16)$$

where all derivatives are taken at $(Q,L) \in \mathfrak{M}_0^{\mathrm{reg}}$. By construction, these formulae give two compatible Poisson brackets on $\mathrm{Hol}(\mathfrak{M})_{\mathrm{red}}$. The same formulae give Poisson algebra structures on $\mathrm{Hol}(\mathfrak{M}_0^{\mathrm{reg}})^{\mathcal{N}}$ and on $\mathrm{Hol}(\mathfrak{M}_0^{\mathrm{reg}})^{G_0}$ as well.

Here, $\mathcal{R}(Q) \in \mathrm{End}(\mathcal{G})$ is the standard trigonometric solution of the modified classical dynamical Yang–Baxter equation. By writing $Q = e^q$ with $q \in \mathcal{G}_0$, for any $X \in \mathcal{G}$ we have

$$(\mathcal{R}(Q)X)_{ii} = 0, \qquad (\mathcal{R}(Q)X)_{ij} = \frac{1}{2} X_{ij} \coth \frac{q_i - q_j}{2}, \ \text{for } i \neq j.$$

How the reduced Poisson bracket formulas were derived?

Basic lemma. Consider $f \in \mathrm{Hol}(\mathfrak{M}_0^{\mathrm{reg}})^{\mathcal{N}}$ given by $f = F \circ \iota$, where $F \in \mathrm{Hol}(\mathfrak{M}^{\mathrm{reg}})^G$. Then the derivatives of f and F satisfy the following relations at any $(Q, L) \in \mathfrak{M}_0^{\mathrm{reg}}$:

$$d_2 F(Q,L) = d_2 f(Q,L), \qquad [L, d_2 f(Q,L)]_0 = 0,$$

$$\nabla_1 F(Q,L) = \nabla_1 f(Q,L) - (\mathcal{R}(Q) + \frac{1}{2}\mathrm{id})[L, d_2 f(Q,L)].$$

The first equalities hold since f is the restriction of F. In particular, it satisfies

$$0 = \frac{d}{dz}\Big|_{z=0} f(Q, e^{zX_0} L e^{-zX_0}) = \langle d_2 f(Q,L), [X_0, L]\rangle = \langle [L, d_2 f(Q,L)], X_0\rangle, \quad \forall X_0 \in \mathcal{G}_0.$$

Next, use the orthogonal decomposition $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_\perp$, and note that the equality of the $\mathcal{G}_0$ parts, $(\nabla_1 F(Q,L))_0 = (\nabla_1 f(Q,L))_0$, is obvious. Then, using any $X \in \mathcal{G}_\perp$, notice that

$$0 = \frac{d}{dz}\Big|_{z=0} F(e^{zX} Q e^{-zX}, e^{zX} L e^{-zX}) = \langle X, (\mathrm{id} - \mathrm{Ad}_{Q^{-1}})\nabla_1 F(Q,L) + [L, d_2 F(Q,L)]\rangle,$$

with $\mathrm{Ad}_Q(Y) := QYQ^{-1}$. Therefore

$$(\mathrm{Ad}_{Q^{-1}} - \mathrm{id})(\nabla_1 F(Q,L))_\perp = [L, d_2 F(Q,L)]_\perp = [L, d_2 f(Q,L)]_\perp.$$

This implies the formula of $(\nabla_1 F(Q,L))_\perp$ by elementary identities. For any $X \in \mathcal{G}_\perp$, one has $(\mathcal{R}(Q) + \frac{1}{2}\mathrm{id})X = \left(\mathrm{id} - \mathrm{Ad}_{Q^{-1}}\right)^{-1}_{|\mathcal{G}_\perp} X$; and $[L, d_2 f(Q,L)] \equiv \nabla_2 f(Q,L) - \nabla_2' f(Q,L)$.

We associate vector fields to the elements of $\mathrm{Hol}(\mathfrak{M})_{\mathrm{red}}$ using the reduced Poisson brackets. In particular, the reduced Hamiltonians

$$h_m := H_m \circ \iota \in \mathrm{Hol}(\mathfrak{M})_{\mathrm{red}}, \quad h_m(Q, L) = \frac{1}{m}\mathrm{tr}(L^m),$$

give rise to the vector fields Y_m^i on $\mathfrak{M}_0^{\mathrm{reg}}$ that satisfy

$$Y_m^i[f] = \{f, h_m\}_i^{\mathrm{red}}, \quad \forall f \in \mathrm{Hol}(\mathfrak{M})_{\mathrm{red}}, \quad i = 1, 2.$$

These vector fields are not unique, since one may add any vector field to Y_m^i that is tangent to the orbits of the residual gauge transformations.

Proposition 7. For all $m \in \mathbb{N}$, the 'reduced Hamiltonian vector fields' Y_m^i can be specified by the formulae

$$Y_{m+1}^1[Q] = Y_m^2[Q] = (L^m)_0 Q, \qquad Y_{m+1}^1[L] = Y_m^2[L] = [\mathcal{R}(Q)L^m, L].$$

Thus, Poisson reduction led to the reduced bi-Hamiltonian evolution equations

$$\dot{Q} = (L^m)_0 Q, \quad \dot{L} = [\mathcal{R}(Q)L^m, L], \quad \text{up to residual gauge transformations.}$$

The standard Lax matrix of the spin Sutherland model is $L = p + (\mathcal{R}(Q) + \frac{1}{2}\mathrm{id})(\phi)$, where p is an arbitrary diagonal and ϕ is an arbitrary off-diagonal matrix. The reduced first Poisson bracket reproduces the standard spin Sutherland Poisson structure of the Q, p, ϕ variables. Indeed, the diagonal entries p_j of p and q_j in $Q_j = e^{q_j}$ form canonically conjugate pairs with respect to the reduced first Poisson bracket, and the vanishing of the diagonal part of ϕ represents a constraint on $\mathfrak{gl}(n, \mathbb{C})^*$ that is responsible for the gauge transformations by G_0. (Here, we refer to $\mathrm{Hol}(\mathfrak{M}_0^{\mathrm{reg}})^{G_0}$.) The Hamiltonians $\mathcal{H}_{m+1}(q, p, \phi) = \frac{1}{m+1}\mathrm{tr}(L^{m+1})$ generate the above evolution equations.

One can obtain bi-Hamiltonian structures for the hyperbolic and trigonometric real forms by restricting L to be Hermitian and q in $Q = e^q$ to be real or purely imaginary, respectively. In the trigonometric case, by further restricting L to the open subset of positive matrices and using a different parametrization, the second Poisson structure becomes identified with that of a spin Ruijsenaars–Schneider model. The trigonometric real form also arises from reduction of $T^*\mathrm{U}(n)$, and the hyperbolic real form from $T^*\mathrm{GL}(n,\mathbb{C})_{\mathbb{R}}$. One may also take L and q to be real, which arises from an open submanifold of $T^*\mathrm{GL}(n,\mathbb{R})$.

References for the reported work

L.F.: Bi-Hamiltonian structure of a dynamical system introduced by Braden and Hone, Nonlinearity 32, 4377-4394 (2019)

L.F.: Reduction of a bi-Hamiltonian hierarchy on $T^*\mathrm{U}(n)$ to spin Ruijsenaars–Sutherland models, Lett. Math. Phys. 110, 1057-1079 (2020)

L.F. and I. Marshall: On the bi-Hamiltonian structure of the trigonometric spin Ruijsenaars–Sutherland hierarchy, pp. 75-87 in: Geometric Methods in Physics XXXVIII, eds. P. Kielanowski et al (Birkhauser, 2020)

L.F.: Bi-Hamiltonian structure of spin Sutherland models: the holomorphic case, Ann. Henri Poincaré 22, 4063-4085 (2021)

A bi-Hamiltonian structure on $\mathfrak{gl}(n,\mathbb{R})$ from Poisson reduction

Let us equip $\mathcal{G} := \mathfrak{gl}(n,\mathbb{R})$ with the trace form, and consider its vector space decomposition $\mathcal{G} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A} := \mathrm{o}(n,\mathbb{R})$ and $\mathcal{B}$ is the upper-triangular subalgebra. Then

$$R = \frac{1}{2}(\pi_{\mathcal{B}} - \pi_{\mathcal{A}})$$

is a solution of the modified classical Yang–Baxter equation. Its has the anti-symmetric and symmetric parts R_a and R_s, and R_a solves the same equation as R. Actually, $R_a \equiv r = \frac{1}{2}(\pi_> - \pi_<)$ is the real version of the r-matrix used before.

As an example of results of Li–Parmentier and Oevel–Ragnisco from 1989, the following formulas define compatible Poisson brackets on $\mathfrak{gl}(n,\mathbb{R})$:

$$\{f,h\}_2 := \langle \nabla f, R_a \nabla h\rangle - \langle \nabla' f, R_a \nabla' h\rangle + \langle \nabla f, R_s \nabla' h\rangle - \langle \nabla' f, R_s \nabla h\rangle,$$

$$\{f,h\}_1(L) = \langle L, [R df(L), dh(L)] + [df(L), R dh(L)]\rangle.$$

Here $\nabla f(L) := L df(L)$ and $\nabla' f(L) := df(L) L$. The Hamiltonians $h_k(L) := \frac{1}{k}\mathrm{tr}(L^k)$ $(k \in \mathbb{N})$ enjoy the relation

$$\{f, h_k\}_2 = \{f, h_{k+1}\}_1, \qquad \forall f \in C^\infty(\mathcal{G}),$$

and their Hamiltonian vector fields engender bi-Hamiltonian Lax equations:

$$\partial_{t_k}(L) := \{L, h_k\}_2 = \{L, h_{k+1}\}_1 = [R(L^k), L], \qquad \forall k \in \mathbb{N}.$$

It is known that the symmetric matrices as well as the tri-diagonal symmetric matrices form Poisson submanifolds for both brackets. Taking the tri-diagonal Jacobi matrices, one recovers the bi-Hamiltonian structure of the open Toda lattice.

The linear r-matrix bracket given above is well known to descend by Poisson reduction from the cotangent bundle of $G = \mathrm{GL}(n, \mathbb{R})$:

$$T^*G \simeq \mathfrak{M} := G \times \mathcal{G} = \{(g, L) \mid g \in G,\ L \in \mathcal{G}\}.$$

In a recent short note, we have shown that the aforementioned quadratic bracket also descends from $\mathfrak{M}$. For this, we first equip $\mathfrak{M}$ with the two compatible Poisson brackets defined as before, but taking everything real instead of complex.

We consider those functions on $\mathfrak{M}$ that are invariant with respect to the symmetry group $S := A \times B$, where $A := \mathrm{O}(n, \mathbb{R})$ and B consists of the upper triangular elements of G having positive diagonal entries. The action of S on $\mathfrak{M}$ is given by letting any $(a, b) \in A \times B$ act on $(g, L) \in \mathfrak{M}$ by the diffeomorphism $(g, L) \mapsto (agb^{-1}, aLa^{-1})$.

Thanks to the QU factorization (Gram–Schmidt) we may associate with any smooth, S-invariant functions F, H on $\mathfrak{M}$ unique smooth functions f, h on $\mathcal{G}$ according to the rule

$$f(L) := F(\mathbf{1}_n, L), \quad h(L) := H(\mathbf{1}_n, L).$$

The invariant functions turn out to close under both Poisson brackets on $\mathfrak{M}$, and thus we may define the reduced Poisson brackets on $C^\infty(\mathcal{G})$ by setting

$$\{f, h\}_i^{\mathrm{red}}(L) := \{F, H\}_i(\mathbf{1}_n, L), \qquad i = 1, 2.$$

These reduced Poisson brackets reproduce the ones displayed on the preceding slide.

For details, see L.F. and B. Juhász: A note on quadratic Poisson brackets on $\mathfrak{gl}(n, \mathbb{R})$ related to Toda lattices, Lett. Math. Phys. 112:45 (2022)

Finally, we outline the derivation of bi-Hamiltonian models, whose main Hamiltonian 'in physical variables' reads

$$\mathcal{H}_{\text{spin}-2} = \frac{1}{2}\sum_{i=1}^{n}(p_i^2 - |\xi_{ii}^l|^2) + \sum_{1\leq i<j\leq n}\left(\frac{|\xi_{ij}^l|^2 + |\xi_{ij}^r|^2 - 2\Re(\xi_{ij}^r\xi_{ji}^l)}{\sinh^2(q_i - q_j)} + \frac{\Re(\xi_{ij}^r\xi_{ji}^l)}{\sinh^2((q_i - q_j)/2)}\right).$$

The two spins $\xi^l, \xi^r \in \mathfrak{u}(n)^* \simeq \mathfrak{u}(n)$ are coupled by the constraint that the diagonal part of $(\xi^l + \xi^r)$ vanishes, and they matter up to $\mathbb{T}^n$ gauge transformations that act on them by simultaneous conjugations. Here q and p are real, and upon setting $\xi^l = 0$ we recover the hyperbolic spin Sutherland Hamiltonian.

Our starting point is the cotangent bundle of the real Lie group $G := \mathrm{GL}(n, \mathbb{C})_{\mathbb{R}}$, with Lie algebra $\mathcal{G} := \mathfrak{gl}(n, \mathbb{C})_{\mathbb{R}}$, which we identify with

$$\mathfrak{M} = G \times \mathcal{G} = \{(g, J) \mid g \in G,\ J \in \mathcal{G}\},$$

where the real pairing $\langle X, Y\rangle = \Re\mathrm{tr}(XY)$ is used on $\mathcal{G} \simeq \mathcal{G}^*$. Then the real manifold $\mathfrak{M}$ carries a bi-Hamiltonian structure given by the same formulas as in the holomorphic case, but using this real-valued pairing. For $F, H \in C^\infty(\mathfrak{M}, \mathbb{R})$:

$$\{F, H\}_1(g, J) = \langle \nabla_1 F, d_2 H\rangle - \langle \nabla_1 H, d_2 F\rangle + \langle J, [d_2 F, d_2 H]\rangle,$$

$$\begin{aligned}\{F, H\}_2(g, J) \;=\; & \langle r\nabla_1 F, \nabla_1 H\rangle - \langle r\nabla_1' F, \nabla_1' H\rangle \\ & + \langle \nabla_2 F - \nabla_2' F, r_+\nabla_2' H - r_-\nabla_2 H\rangle \\ & + \langle \nabla_1 F, r_+\nabla_2' H - r_-\nabla_2 H\rangle - \langle \nabla_1 H, r_+\nabla_2' F - r_-\nabla_2 F\rangle.\end{aligned}$$

For any $k \in \mathbb{N}$, we have the 'free Hamiltonians' H_k and $\tilde{H}_k$ on $\mathfrak{M}$ defined by

$$H_k(g,J) := \frac{1}{k}\Re\mathrm{tr}(J^k), \qquad \tilde{H}_k(g,J) := \frac{1}{k}\Im\mathrm{tr}(J^k).$$

All these Hamiltonians are in involution, and they define bi-Hamiltonian systems according to the relations, and corresponding flows, listed as follows:

$$\{\,\cdot\,,H_k\}_2 = \{\,\cdot\,,H_{k+1}\}_1, \qquad (g(t),J(t)) = \left(\exp\left(J(0)^k t\right) g(0), J(0)\right),$$

$$\{\,\cdot\,,\tilde{H}_k\}_2 = \{\,\cdot\,,\tilde{H}_{k+1}\}_1, \qquad (g(t),J(t)) = \left(\exp\left(-\mathrm{i}J(0)^k t\right) g(0), J(0)\right).$$

We consider the symmetry group $\mathrm{U}(n) \times \mathrm{U}(n)$ acting on $\mathfrak{M}$ by the diffeomorphisms

$$A_{\eta_L,\eta_R} : (g,J) \mapsto (\eta_L g \eta_R^{-1}, \eta_L J \eta_L^{-1}).$$

The invariant functions close under both Poisson brackets, and thus we obtain a bi-Hamiltonian structure on the quotient space $\mathfrak{M}_{\mathrm{red}} = \mathfrak{M}/(\mathrm{U}(n) \times \mathrm{U}(n))$, whose space of smooth functions is $C^\infty(\mathfrak{M})^{\mathrm{U}(n)\times\mathrm{U}(n)}$.

We describe the reduced Poisson algebras by using the singular value decomposition, whereby every $g \in \mathrm{GL}(n,\mathbb{C})$ can be decomposed as

$$g = \eta_L e^q \eta_R^{-1}, \quad \eta_L, \eta_R \in \mathrm{U}(n), \quad q = \mathrm{diag}(q_1, q_2, \dots, q_n), \quad q_i \in \mathbb{R},\ q_1 \ge q_2 \ge \cdots \ge q_n.$$

Every invariant function $F \in C^\infty(\mathfrak{M})^{\mathrm{U}(n)\times\mathrm{U}(n)}$ can be recovered from its restriction, f, to the following submanifold of $\mathfrak{M}$:

$$\mathfrak{M}_0^{\mathrm{reg}} := \{(e^q, J) \mid J \in \mathcal{G},\ q = \mathrm{diag}(q_1, q_2, \ldots, q_n),\ q_1 > q_2 > \cdots > q_n\}.$$

The $\mathrm{U}(n) \times \mathrm{U}(n)$ orbits through $\mathfrak{M}_0^{\mathrm{reg}}$ fill the dense open submanifold $\mathfrak{M}^{\mathrm{reg}}$, and we get

$$C^\infty(\mathfrak{M}^{\mathrm{reg}})^{\mathrm{U}(n)\times\mathrm{U}(n)} \Longleftrightarrow C^\infty(\mathfrak{M}_0^{\mathrm{reg}})^{\mathbb{T}^n}.$$

The Poisson brackets on $C^\infty(\mathfrak{M}^{\mathrm{reg}})^{\mathrm{U}(n)\times\mathrm{U}(n)}$ translate into the reduced PBs $\{f, h\}_i^{\mathrm{red}}$ on $C^\infty(\mathfrak{M}_0^{\mathrm{reg}})^{\mathbb{T}^n}$.

We derived the form of the compatible reduced Poisson brackets. Then we recovered the Sutherland model coupled to two spins by applying a suitable parametrization to the first reduced Poisson bracket. Namely, we parametrize $J \in \mathcal{G}$ as follows:

$$J_{ij} = p_i \delta_{ij} - (1 - \delta_{ij}) \left(\coth(q_i - q_j)\xi^l_{ij} + \xi^r_{ij}/\sinh(q_i - q_j)\right) - \xi^l_{ij}, \qquad \forall 1 \leq i, j \leq n,$$

with $\xi^l, \xi^r \in \mathrm{u}(n)$, satisfying $\xi^l_{ii} + \xi^r_{ii} = 0$.

For these results, see L.F.: Bi-Hamiltonian structure of Sutherland models coupled to two $\mathfrak{u}(n)^*$-valued spins from Poisson reduction, Nonlinearity 35, 2971-3003 (2022). There one can find references to earlier papers (by L.F.–Pusztai, Kharchev–Levin–Olshanetsky–Zotov, Reshetikhin) devoted to similar models, but the bi-Hamiltonian aspects were not studied before.

Conclusion

We observed that the cotangent bundles

$$T^*\mathrm{GL}(n,\mathbb{C}), \quad T^*\mathrm{U}(n), \quad T^*\mathrm{GL}(n,\mathbb{C})_{\mathbb{R}}, \quad T^*\mathrm{GL}(n,\mathbb{R})$$

carry natural bi-Hamiltonian structures.

Then we have shown that the Poisson reduction procedures that were studied before using the canonical Poisson bracket equip the reduced systems with bi-Hamiltonian structures. The interpretation of the reduced systems as spin Sutherland models relies on the reduced canonical Poisson bracket (and similar for the open Toda system).

All the reduced systems that we obtained are 'strongly expected' to possess the property of degenerate integrability on generic symplectic leaves in the (full, stratified) reduced phase space.

How to find bi-Hamiltonian structures for elliptic spin Sutherland models?