Celestial mechanics: The perturbed Kepler problem.

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January 14, 2015

Completed with the support of the European Union and the State of Hungary, cofinanced by the European Social Fund in the framework of TÁMOP 4.2.4. A/2-11-/1-2012-0001 'National Excellence Program'

Contents

1 Introduction

In Chapter 2 we review the basics of Keplerian motion in a form suitable for generalization to the perturbed case. The solution of the two-body problem is given both in terms of dynamical constants and orbital elements, however the accent is put on the former, as this is what we want to explore in more detail in the perturbed case.

Chapter 3 contains the generic discussion of the perturbed two-body problem. In some cases the perturbation can be related to a generalized Lagrangian (possibly depending on accelerations). We show how to derive the acceleration from such a Lagrangian. Then in the rest of the chapter we develop the equations governing the slow evolution of the Newtonian constants and of the reference system constructed from them, in terms of the components of the perturbative force (irrespective of whether these arise or not from a Lagrangian). The evolving Newtonian constants represent a sequence of Keplerian orbits with different parameters, which match well the perturbed orbit at each point. The Keplerian orbit with varying parameters represents the osculating orbit, a notion usually employed for the ellipse.

Chapter 4 relates the variation in time of the Keplerian dynamical constants to the evolution of the orbital elements. The latter are known as the Lagrange planetary equations, equivalent with the evolution equations for the osculating dynamical constants, provided a reference plane and a reference direction are singled out.

In chapter 5 the osculating dynamical constants are derived explicitly in terms of the true anomaly parametrization, which can be introduced exactly as in the unperturbed case, due to a radial equation, which has an unmodified functional form as compared to the unperturbed equation. The functional form of the evolution law for the true anomaly however is modified.

In chapter 6 the eccentric anomaly is introduced exactly in the same way as in the unperturbed case (same functional form). Its evolution is derived explicitly and by a formal integration the dynamical law governing the perturbed motion, the perturbed Kepler equation, is established. In the process the osculating dynamical constants are determined as function of the eccentric anomaly, the radial period of the perturbed motion is defined and the condition of circularity of perturbed Keplerian orbits is analyzed.

As a first application of the formalism derived, we consider a constant perturbing force in chapter 7. Such a force can possibly act on a binary system due to a distant supermassive black hole. This computationally simple toy model allows to explicitly perform all calculations presented only formally in the previous chapters. The osculating dynamical constants are determined, the periastron shift computed and the perturbed Kepler equation written up explicitly. Another application is proposed as a problem to be developed by the students.

The gravitational constant G is kept in all expressions. A vector with an overhat denotes a unit vector, the only exception under this rule being $\mathbf{n} = \mathbf{r}/r$. Time derivatives are denoted both by a dot or by d/dt.

2 Keplerian motion

The motion of two point masses under their mutual gravitational attraction is characterized by the one-center Lagrangian:

$$\mathcal{L}_N = \frac{\mu v^2}{2} + \frac{Gm\mu}{r} \ , \tag{1}$$

where $m = m_1 + m_2$ is the total mass, $\mu = m_1 m_2/m$ the reduced mass, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 = r\mathbf{n}$, with r the relative distance and v the magnitude of the relative velocity $\mathbf{v} = \mathbf{p}_N/\mu$, $\mathbf{p}_N = \partial\mathcal{L}_N/\partial\mathbf{v}$ being the relative momentum). The Lagrangian represents a particle of mass μ moving in the gravitational potential of a fixed mass M. The Euler-Lagrange equations give the Newtonian acceleration

$$\mathbf{a}_N \equiv \frac{1}{\mu}\frac{d}{dt}\mathbf{p}_N = \frac{1}{\mu}\frac{\partial}{\partial\mathbf{r}}\mathcal{L}_N = -\frac{Gm}{r^2}\mathbf{n} \ . \tag{2}$$

2.1 The one-center problem

The Lagrangian (1) describes the so-called one-center problem: a particle of mass μ orbiting a mass m fixed in the origin. It is obtained from the two-body problem with dynamical equations

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= \frac{Gm_1}{r^3}\mathbf{r} \ , \\ \ddot{\mathbf{r}}_2 &= -\frac{Gm_2}{r^3}\mathbf{r} \ . \end{aligned} \tag{3}$$

These equations arise from the two-body Lagrangian

$$\mathcal{L}_N^{2-body} = \frac{m_1\dot{\mathbf{r}}_1^2}{2} + \frac{m_2\dot{\mathbf{r}}_2^2}{2} + \frac{Gm_1m_2}{r} \ . \tag{4}$$

By introducing the center of mass vector $\mathbf{r}_{CM} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/M$, then

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_{CM} - \frac{m_2}{M}\mathbf{r} \ , \\ \mathbf{r}_1 &= \mathbf{r}_{CM} + \frac{m_1}{M}\mathbf{r} \ , \end{aligned} \tag{5}$$

and the two-body Lagrangian becomes

$$\mathcal{L}_N^{2-body} = \frac{M\dot{\mathbf{r}}_{CM}^2}{2} + \mathcal{L}_N \ . \tag{6}$$

The dynamics (3) can be rewritten as:

$$\frac{d^2}{dt^2}\mathbf{r} = -\frac{Gm}{r^3}\mathbf{r} \ , \tag{7}$$

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = \mathbf{P}t + \mathbf{K} \ . \tag{8}$$

Eq. (7) is the dynamical equation (2), while Eq. (8) gives the evolution of the center of mass, with the constants $\mathbf{P}$ representing the total momentum of the system, and $\mathbf{K}/m$ the position of the center of mass at $t = 0$. The one-center problem is obtained by transforming to the center of mass system, e.g. by choosing $\mathbf{P} = \mathbf{K} = 0$. With this choice the two-body and one-center Lagrangians coincide.

We note that according to the general N-body theory the differential order of the dynamical system is $6N-10$ (given by $3N$ second order Newtonian equations for the three coordinates of each particle, minus 7 constants of motion: the energy, momentum and angular momentum, minus 3 scalars giving the position of the center of mass). For the two-body system the differential order is then 2, which means that we have to integrate twice in order to solve the problem.

When we reduce the two-body problem to the one-center problem, we go to the center of mass system, reducing the differential order by 6 (three positions and three momenta being fixed). Therefore the differential order becomes 6, further reduced by 4 by the remaining constants of motion: the orbital angular momentum vector and the energy, such that there are still 2 integrations to be performed.

2.2 Constants of motion

The shape of a Keplerian orbit and orientation of the orbital plane are completely determined by the energy

$$E_N \equiv \mathbf{v} \cdot \mathbf{p}_N - \mathcal{L}_N = \left[-1 + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{v}} \right] \mathcal{L}_N = \frac{\mu v^2}{2} - \frac{Gm\mu}{r} \ , \tag{9}$$

and orbital angular momentum

$$\mathbf{L_N} \equiv \mathbf{r} \times \mathbf{p}_N = \left[\mathbf{r} \times \frac{\partial}{\partial \mathbf{v}} \right] \mathcal{L}_N = \mu \mathbf{r} \times \mathbf{v} \ . \tag{10}$$

It is straightforward to check directly that both are conserved.[1] There is another constant of the motion, the Laplace-Runge-Lenz vector

$$\mathbf{A_N} \equiv \mathbf{v} \times \mathbf{L_N} - \frac{Gm\mu}{r}\mathbf{r} \ , \tag{11}$$

which satisfies the constraints

$$A_N^2 = \frac{2E_N L_N^2}{\mu} + (Gm\mu)^2 \ , \tag{12}$$

and

$$\mathbf{L_N} \cdot \mathbf{A_N} = 0 \ . \tag{13}$$

[1] In contrast with the N-body problem, where the momentum, angular momentum and energy are all constants of the motion, in the *reduced* 2-body problem (one-center problem) the momentum $\mathbf{p}_N = \partial \mathcal{L}_N / \partial \mathbf{v} = \mu \mathbf{v}$ of the reduced mass particle μ orbiting the fixed total mass m is obviously not a constant.

Due to the constraint (12) only one of its components is independent of the other constants of motion $E_N, \mathbf{L_N}$. Thus there are five constants of motion altogether, which means that there is one integration left in order to solve the problem. Due to the constraint (13) the Laplace-Runge-Lenz vector lies in the plane of motion, determined by $\hat{\mathbf{L}}_\mathbf{N}$ (the direction of $\mathbf{L_N}$). Its magnitude being expressed in terms of E_N and L_N, the only independent information carried by $\mathbf{A_N}$ remains its orientation in the plane of the orbit.

2.3 The orbit

2.3.1 Parametrization of the radial motion by the true anomaly

As the orbital angular momentum is constant, the plane of motion is conserved. Eqs. (9) and (10) imply that Newtonian dynamics is determined by E_N and L_N, as encoded in the relations:

$$v^2 = \frac{2E_N}{\mu} + \frac{2Gm}{r} \ , \tag{14}$$

$$\dot{\psi} = \frac{L_N}{\mu r^2} \ . \tag{15}$$

Here ψ is the azimuthal angle in the fixed plane of motion. As consequence of $v^2 = \dot{r}^2 + r^2\dot{\psi}^2$, a first order differential equation for the radial variable $r\,(t)$ (the radial equation) is found:

$$\dot{r}^2 = \frac{2E_N}{\mu} + \frac{2Gm}{r} - \frac{L_N^2}{\mu^2 r^2} \ . \tag{16}$$

Eqs. (15) and (16) give

$$\frac{d\left(\frac{Gm\mu}{A_N} - \frac{L_N^2}{\mu A_N r}\right)}{\sqrt{1-\left(\frac{Gm\mu}{A_N} - \frac{L_N^2}{\mu A_N r}\right)^2}} = \pm d\psi \ , \tag{17}$$

with the $+$ $(-)$ sign applying when r increases (decreases) in time. With the change of variable

$$r = \frac{L_N^2}{\mu\,(Gm\mu + A_N\cos\chi)} \ , \tag{18}$$

Eq. (17) becomes $\mathrm{sgn}\,(\sin\chi)\,d\chi = \pm d\psi$, solved for

$$\chi = \psi - \psi_0 \ , \tag{19}$$

the *true anomaly* angle χ being 0 at the point of closest approach (the periastron), where the azimuthal angle is ψ_0 and $r = r_{\min}$. This choice gives the correct sign both when the point masses approach each other (at $\chi < 0$), and when the separation increases (at $\chi > 0$). The most distant point of the orbit is found for $\chi = \pm\pi$ if $Gm\mu \geq A_N$ (such that r stays positive). If the inequality is strict, this represents the apastron and the orbit is bounded ($r = r_{\max}$ is finite at $\chi = \pm\pi$). For $Gm\mu = A_N$ the orbit opens up with $r \to \infty$, when $\chi \to \pm\pi$. For $Gm\mu < A_N$ the orbit becomes unbounded ($r \to \infty$) already for certain $\chi_\pm$ with $|\chi_\pm| < \pi$.

As discussed in Section 2.2, there are 5 constants of motion in the one-center problem, therefore the differential order is $6 - 5 = 1$. Although obtained in an integration process, the expressions (18) and (19) represent only a new radial variable (which happens to be an angle), and we still have to perform the integration giving the complete solution of the problem. The remaining differential equation is Eq. (15). By employing the true anomaly, this can be rewritten as the coupled system:

$$\dot{r} = \frac{A_N}{L_N} \sin\chi \ , \tag{20}$$

$$\frac{dt}{d\chi} = \frac{\mu}{L_N} r^2 \ . \tag{21}$$

Eq. (20) shows that there are turning points ($\dot{r} = 0$) only at $\chi = 0, \pm\pi$. Before carrying on the remaining integration, we discuss in more detail the Keplerian orbits.

2.3.2 Conics

The orbit $r(\psi)$, as given by Eqs. (18) and (19), represents a conic section, the intersection of a plane with a cone. We can see this by introducing the parameter (semilatus rectum) and the eccentricity as

$$p = \frac{L_N^2}{Gm\mu^2} \ , \tag{22}$$

$$e = \frac{A_N}{Gm\mu} \ . \tag{23}$$

Eq. (23) relates the magnitude of the Laplace-Runge-Lenz vector to the eccentricity of the orbit. Due to the constraint (12) the eccentricity can be also expressed in terms of $E_N,\ L_N$. According to our previous remarks, $e \in [0, 1)$ characterizes bounded orbits, while $e \geq 1$ is for unbounded orbits.

Eq. (18) becomes

$$r = \frac{p}{1 + e\cos\chi} \ . \tag{24}$$

We can get further information on these orbits by rewriting the equation (24) into Cartesian coordinates $x = r\cos\chi,\ y = r\sin\chi$. We get:

$$\left(1 - e^2\right) x^2 + y^2 = p^2 - 2epx \ , \qquad \text{with } ex < p \ . \tag{25}$$

When $x = 0$, the above equation gives $y = \pm p$, the points of intersection of the conic with the y-axis. There are four distinct classes of conics, distinguished by the value of eccentricity. Eq. (25) describes:

$$\text{a circle } (e=0): \qquad x^2+y^2=p^2 \ , \tag{26}$$

$$\text{an ellipse } (0<e<1): \qquad \frac{\left(x+\frac{ep}{1-e^2}\right)^2}{\left(\frac{p}{1-e^2}\right)^2}+\frac{y^2}{\left(\frac{p}{\sqrt{1-e^2}}\right)^2}=1 \ , \tag{27}$$

$$\text{a parabola } (e=1): \qquad y^2=p^2-2px \ , \tag{28}$$

$$\text{a hyperbola } (e>1): \qquad \frac{\left(\frac{ep}{e^2-1}-x\right)^2}{\left(\frac{p}{e^2-1}\right)^2}-\frac{y^2}{\left(\frac{p}{\sqrt{e^2-1}}\right)^2}=1 \ . \tag{29}$$

All these curves are represented in the center of mass system (see Fig. 1), the origin being the focus of the conics.

The orbit with $e=0$ represents a circle with radius p.

The orbits with $e\in(0,1)$ are ellipses. Eq. (27) gives their semimajor and semiminor axes as $a=p/\left(1-e^2\right)$ and $b=p/\sqrt{1-e^2}$. The distances of the apastron and periastron measured from the focus are found either from Eq. (24) for $\chi=\pi,0$ as $r_{\substack{\max\\\min}}=p/(1\mp e)=a(1\pm e)$ or from Eq. (27) for $y=0$ as $x=\mp r_{\substack{\max\\\min}}$. The distance between the focus and the center of symmetry of the ellipse is $a-r_{\min}=ae$ (the linear eccentricity).

For $e=1$ Eq. (24) represents a parabola and for $e\in(1,\infty)$ a hyperbola. Their point of closest approach to the focus is given by $r_{\min}=p/2$ for the parabola and $r_{\min}=p/(1+e)$ for the hyperbola. Both are unbounded orbits with $\lim_{r\to\infty}\chi=\pm\pi$ for the parabola and $\lim_{r\to\infty}\chi=\chi_{\pm}=\pm\arccos(-1/e)$ for the hyperbola. This means that at infinity the two branches of the parabola become parallel (and horizontal), while the two branches of the hyperbola asymptote to the directions $\chi_{\pm}$. For $e\to\infty$ the hyperbola becomes a straight line, with $\chi_{\pm}^{\infty}=\pm\pi/2$. In analogy with the ellipse, for the hyperbola we define $a=p/\left(e^2-1\right)$ and $b=p/\sqrt{e^2-1}$. Then for both the ellipse and hyperbola the parameter can be expressed as $p=b^2/a$.

By shifting horizontally all points of a conic with a distance which is e^{-1} times their distance from the focus, we obtain $x_d=e^{-1}r+r\cos\chi=p/e=$constant, which defines a vertical line, the directrix. The constant x_d is also known as the focal parameter. The focus, the directrix and the eccentricity represent an equivalent definition of the conics (the circle being an exception). For an ellipse, the distance between the symmetry center and the directrix is $p/e+ae=a/e$. (The circular orbit limit $p=a=b$ becomes degenerated: for a given radius and center the directrix is shifted to infinity; while for a given directrix and center at finite separation the radius becomes zero.)

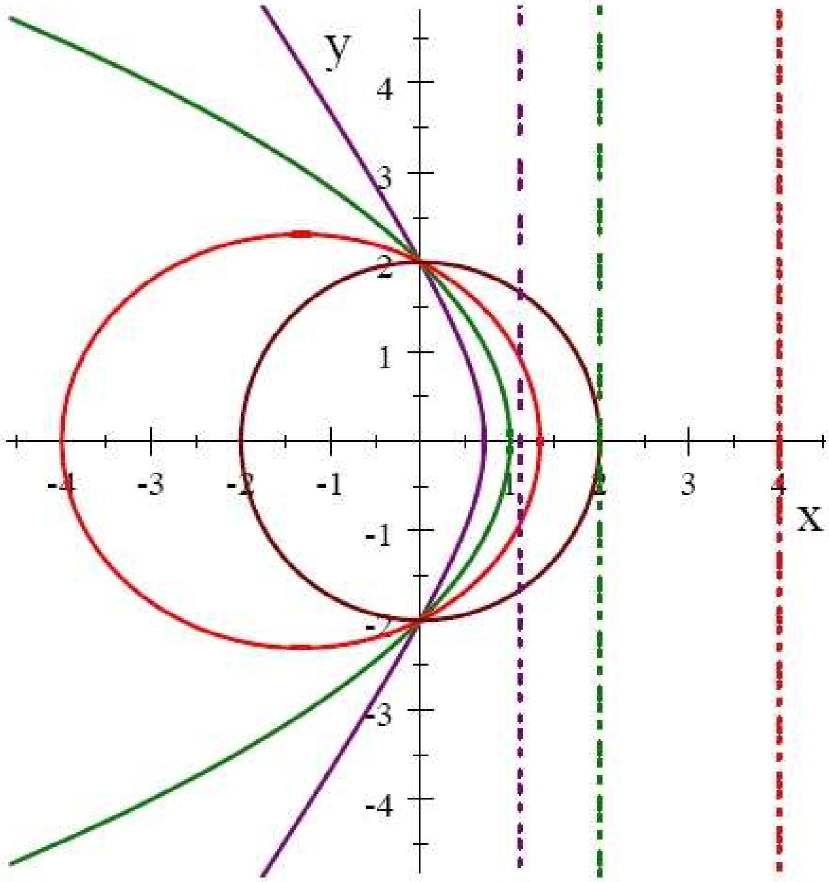

Figure 1: Circular ($e = 0$; brown), elliptic ($e = 0.5$; red), parabolic ($e = 1$; green), hyperbolic ($e = 1.8$; purple) orbits and the directrices for the ellipse, parabola and hyperbola, represented in the center of mass system (the origin is in the focus of the conics). The parameter (here $p = 2$) defines the intersection points of the conics with the y-axis. While the circle and the ellipse run over the whole allowed domain $\chi \in (-\pi, \pi]$, the parabola is depicted only for the range $\chi \in [-2.35, 2, 35]$ and the hyperbola for $\chi \in [-1.92, 1.92]$. All represented orbits have the same orbital angular momentum, but different energy $E_N^{circular} < E_N^{elliptic} < E_N^{parabolic} = 0 < E_N^{hyperbolic}$.

2.3.3 Dynamical characterization of the periastron

For non-circular orbits we introduce the following orthogonal basis in the plane of motion, constructed from the constants of motion:

$$\begin{aligned} \mathbf{A_N} &= \mu\left(\frac{2E_N}{\mu}+\frac{Gm}{r}\right)\mathbf{r}-\mu r\dot{r}\mathbf{v}\ , \\ \mathbf{Q_N} &= \mathbf{L_N}\times\mathbf{A_N} = Gm\mu^2\dot{r}\mathbf{r}+\left(L_N^2-Gm\mu^2 r\right)\mathbf{v}\ . \end{aligned} \tag{30}$$

Thus for generic orbits the unit vectors

$$\left\{\mathbf{f_{(i)}}\right\} = (\mathbf{\hat{A}_N},\ \mathbf{\hat{Q}_N},\ \mathbf{\hat{L}_N}) \tag{31}$$

form a fixed orthonormal basis. The position and velocity vectors in this basis are:

$$\mathbf{r} = \frac{1}{\mu}\left(\rho\mathbf{\hat{A}_N}+\sigma\mathbf{\hat{Q}_N}\right), \tag{32}$$

$$\mathbf{v} = \tau\mathbf{\hat{A}_N}+\lambda\mathbf{\hat{Q}_N}\ , \tag{33}$$

with coefficients

$$\begin{aligned} \rho &= \frac{L_N^2-Gm\mu^2 r}{A_N}\ , \qquad \sigma = \frac{L_N}{A_N}\mu r\dot{r}\ , \\ \tau &= -\frac{Gm\mu}{A_N}\dot{r}\ , \qquad \lambda = \frac{L_N}{A_N}\left(\frac{2E_N}{\mu}+\frac{Gm}{r}\right)\ . \end{aligned} \tag{34}$$

[We note that in terms of the coefficients (34): $L_N = \rho\lambda-\sigma\tau$.]

The coefficients (34) can be rewritten in terms of the true anomaly by using Eqs. (12), (18), and (20):

$$\begin{aligned} \rho &= \mu r\cos\chi\ , \qquad \sigma = \mu r\sin\chi\ , \\ \tau &= -\frac{Gm\mu}{L_N}\sin\chi\ , \qquad \lambda = \frac{A_N+Gm\mu\cos\chi}{L_N}\ . \end{aligned} \tag{35}$$

The position vector (32) thus becomes

$$\mathbf{r} = r\left(\mathbf{\hat{A}_N}\cos\chi+\mathbf{\hat{Q}_N}\sin\chi\right)\ . \tag{36}$$

At the turning point(s) of the radial motion the position vector is aligned with the Laplace-Runge-Lenz vector: $\mathbf{r} = r\cos\chi\mathbf{\hat{A}_N}$. At the periastron $\mathbf{r}\,(\chi=0) = r_{\min}\mathbf{\hat{A}_N}$, therefore we conclude that the Laplace-Runge-Lenz vector points toward the periastron.

The basis $\left\{\mathbf{\hat{r}},\ \mathbf{\hat{L}_N}\times\mathbf{\hat{r}}\right\}$ is related to the basis $\left\{\mathbf{\hat{A}_N},\ \mathbf{\hat{Q}_N}\right\}$ by a rotation with angle χ in the plane of motion, which is also obvious from the definition of the true anomaly. In this basis in terms of the true anomaly the velocity is expressed as

$$\mathbf{v}=\frac{Gm\mu}{L_N}\left[-\hat{\mathbf{A}}_{\mathbf{N}}\sin\chi+\hat{\mathbf{Q}}_{\mathbf{N}}\left(\cos\chi+\frac{A_N}{Gm\mu}\right)\right] . \tag{37}$$

In summary $\hat{\mathbf{L}}_{\mathbf{N}}$ gives the plane of motion, two of the three dependent quantities (E_N, L_N, A_N) determine the shape of the orbit, finally $\hat{\mathbf{A}}_{\mathbf{N}}$ indicates the position of the periastron. This completes the characterization of the Keplerian orbit in terms of dynamical constraints.

2.4 Solution of the equations of motion: the Kepler equation

Eq. (21) can be rewritten as [see Eq. (72)]:

$$dt=\frac{2L_N^3\left(1+\tan^2\frac{\chi}{2}\right)d\left(\tan\frac{\chi}{2}\right)}{\mu\left(Gm\mu+A_N\right)^2\left(1+\frac{Gm\mu-A_N}{Gm\mu+A_N}\tan^2\frac{\chi}{2}\right)^2} . \tag{38}$$

We introduce $y=\tan\left(\chi/2\right)$ as a new integration variable. There are three cases to consider, depending on the sign of $(Gm\mu-A_N)$. For circular or elliptic orbits $\alpha=\left(Gm\mu-A_N\right)/\left(Gm\mu+A_N\right)\in(0,1]$, for parabolic orbits $Gm\mu-A_N=0$, while for hyperbolic orbits $\beta\equiv-\alpha=\left(A_N-Gm\mu\right)/\left(A_N+Gm\mu\right)\in(0,1)$. Therefore in each case we evaluate one of the integrals:

$$\int\frac{\left(1+y^2\right)dy}{\left(1+\alpha y^2\right)^2} = \frac{1+\alpha}{2\alpha^{3/2}}\left[\arctan(\alpha^{1/2}y)-\frac{1-\alpha}{1+\alpha}\frac{\alpha^{1/2}y}{(1+\alpha y^2)}\right] , \tag{39}$$

$$\int\left(1+y^2\right)dy = y+\frac{1}{3}y^3 , \tag{40}$$

$$\int\frac{\left(1+y^2\right)dy}{\left(1-\beta y^2\right)^2} = \frac{\beta-1}{2\beta^{3/2}}\left[\operatorname{arctanh}(\beta^{1/2}y)-\frac{1+\beta}{1-\beta}\frac{\beta^{1/2}y}{(1-\beta y^2)}\right] . \tag{41}$$

2.4.1 Circular and elliptic orbits

It is straightforward to introduce the variable

$$x=\sqrt{\frac{Gm\mu-A_N}{Gm\mu+A_N}}\tan\frac{\chi}{2} , \tag{42}$$

in terms of which [employing the result (39)] the remaining dynamical equation, Eq. (38) can be integrated as

$$t-t_0=2Gm\left(\frac{\mu}{-2E_N}\right)^{3/2}\left(\arctan x-\frac{A_N}{Gm\mu}\frac{x}{1+x^2}\right) . \tag{43}$$

With the change of variable

$$x = \tan \frac{\xi}{2} \tag{44}$$

this gives the Kepler equation

$$t - t_0 = Gm \left(\frac{\mu}{-2E_N} \right)^{3/2} \left(\xi - \frac{A_N}{Gm\mu} \sin \xi \right) . \tag{45}$$

The parameter ξ is the *eccentric anomaly*, and as can be seen from its definition, Eqs. (42) and (44) it passes through the values $0, \pm\pi, \pm 2\pi, ...$ together with χ. From $\xi = 0$ to $\xi = 2\pi$ the time elapsed is $t - t_0 = T_N$, thus we find the radial period

$$T_N = 2\pi Gm \left(\frac{\mu}{-2E_N} \right)^{3/2} . \tag{46}$$

2.4.2 Parabolic orbits

By employing Eq. (40), the equation (38) integrates as:

$$t - t_0 = \frac{L_N^3}{2G^2 m^2 \mu^3} \left(\tan \frac{\chi}{2} + \frac{1}{3} \tan^3 \frac{\chi}{2} \right) . \tag{47}$$

This is the analogue of the Kepler equation for parabolic orbits. The motion along each branch takes infinite time: $\lim_{\chi \to \pm\pi} (t - t_0) = \pm\infty$.

2.4.3 Hyperbolic orbits

By employing Eq. (41) and introducing the variable

$$z = \sqrt{\frac{A_N - Gm\mu}{A_N + Gm\mu}} \tan \frac{\chi}{2} = ix , \tag{48}$$

the equation (38) integrates as:

$$t - t_0 = -2Gm \left(\frac{\mu}{2E_N} \right)^{3/2} \left(\operatorname{arctanh} z - \frac{A_N}{Gm\mu} \frac{z}{1 - z^2} \right) . \tag{49}$$

With the further change of variable

$$z = \tanh \frac{\zeta}{2} \tag{50}$$

we obtain the analogue of the Kepler equation for hyperbolic orbits:

$$t - t_0 = -Gm \left(\frac{\mu}{2E_N} \right)^{3/2} \left(\zeta - \frac{A_N}{Gm\mu} \sinh \zeta \right) . \tag{51}$$

	periastron			most distant point		
	$r_{\rm min}$	$\chi_{\rm min}$	$\xi_{\rm min}$	$r_{\rm max}$	$\chi_{\rm max}$	$\xi_{\rm max}$
elliptic orbits	$a\,(1-e)$	0	0	$a\,(1+e)$	$\pm\pi$	$\pm\pi$
parabolic orbits	$\frac{p}{2}$	0	–	∞	$\pm\pi$	–
hyperbolic orbits	$a\,(e-1)$	0	0	∞	$\pm\arccos\left(-\frac{1}{e}\right)$	$-i\,(\pm\infty)$

Table 1: The values of the radial coordinates at the points of closest approach and maximal separation. The circular orbit limit arises at $e\to 0$ from the elliptic orbits, however for circular orbits the definition of these points is ambiguous. For parabolic orbits, the eccentric anomaly parametrization remains undefined.

Again, the travel time to $\chi_{\pm}=\pm\arccos\left(-1/e\right)$ is infinite. In order to show this, first we remark that

$$\tan\frac{\chi_{\pm}}{2}=\pm\sqrt{\frac{e+1}{e-1}}=\pm\sqrt{\frac{A_N+Gm\mu}{A_N-Gm\mu}}\ , \tag{52}$$

such that Eqs. (48) and (50) give $\zeta_{\pm}=2\ {\rm arctanh}\,(\pm 1)=\pm\infty$, irrespective of the value of eccentricity. Thus, $\lim_{\zeta\to\zeta_{\pm}=\pm\infty}(t-t_0)=\pm\infty$.

Remarkably, the equation (51) can be obtained from the Kepler equation (45) derived for circular / elliptic orbits, by the substitution $\xi\to -i\zeta$. The relation between the variable ζ and the true anomaly χ also emerges from Eqs. (42) and (44) with the same substitution.

We summarize these in Table 2.4.3:

2.4.4 The eccentric anomaly: a summary

In the circular and elliptic case we have introduced the eccentric anomaly cf.:

$$\tan\frac{\xi}{2}=\sqrt{\frac{Gm\mu-A_N}{Gm\mu+A_N}}\tan\frac{\chi}{2}\ . \tag{53}$$

There is no parabolic limit for this equation, however it turned out that it can also be employed in the hyperbolic case, with the change $\xi\to -i\zeta$.

There is actually much simpler to obtain the Kepler equation, provided we already define the eccentric anomaly before integration. Passing to the ξ variable in Eq. (18) by straightforward trigonometric transformations[2] gives the eccentric anomaly parametrization of the radial variable

$$r=\frac{Gm\mu-A_N\cos\xi}{-2E_N}\ . \tag{54}$$

Taking its time derivative and employing Eq. (20) gives

$$\frac{-2E_N}{L_N}\sin\chi=\dot{\xi}\sin\xi\ . \tag{55}$$

[2] The most quicker way would be to employ the forthcoming second relation (76).

Then with the help of Eq. (53) we eliminate χ and obtain:[3]

$$r\dot{\xi} = \left(\frac{-2E_N}{\mu}\right)^{1/2} . \tag{56}$$

With $r(\xi)$ given by Eq. (54) and T_N by Eq. (46) this becomes

$$\frac{2\pi}{T_N}\frac{dt}{d\xi} = 1 - \frac{A_N}{Gm\mu}\cos\xi , \tag{57}$$

a relation, which immediately integrates to the Kepler equation (45). In the circular and elliptic cases T_N represents the orbital period.

In terms of the eccentric anomaly the radial motion and time are parametrized thus as given by Eq. (54) and

$$n_N (t - t_0) = \xi - \frac{A_N}{Gm\mu}\sin\xi , \tag{58}$$

where $n_N = 2\pi/T_N$ is the mean motion and t_0 an integration constant, the time of periastron passage. It is also customary to quote the left hand side of the Kepler equation (58) as the mean anomaly

$$\mathcal{M}_N = n_N (t - t_0) , \tag{59}$$

and $\mathcal{M}_0 = -n_N t_0$ the mean anomaly at the epoch.

Eqs. (54)-(58) are also valid for the hyperbolic case, with T_N given by its definition (46) and by replacing $\xi \rightarrow -i\zeta$.

2.5 Orbital elements

When a reference plane (given on Fig. 2 by its normal $\hat{\mathbf{z}}$) and a reference axis $\hat{\mathbf{x}}$ on it are given (these define an orthonormal reference basis), the orientation of the orbit can be characterized by three angles. The intersection of the plane of the orbit and the reference plane defines the node line. The angle subtended by the plane of motion with the reference plane, ι (the inclination); the angle subtended by the reference direction with the ascending node, Ω (the longitude of the ascending node) and the angle between the ascending node and periastron, ω (the argument of the periastron) are the three angular orbital elements. When the origin of time is chosen at the passage on the ascending node, $\omega = \psi_0$. Instead of ω sometimes the longitude of the periastron, $\tilde{\omega} = \Omega + \omega$ is chosen.

The orthonormal basis (31) is expressed in the reference system in terms of the above angles (see Fig. 2) as:

$$\begin{pmatrix} \hat{\mathbf{A}}_\mathbf{N} \\ \hat{\mathbf{Q}}_\mathbf{N} \end{pmatrix} = \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \hat{\mathbf{l}} \\ \hat{\mathbf{m}} \end{pmatrix} , \tag{60}$$

$$\hat{\mathbf{L}}_\mathbf{N} = \begin{pmatrix} \sin\iota\sin\Omega \\ -\sin\iota\cos\Omega \\ \cos\iota \end{pmatrix} , \tag{61}$$

[3]The most quicker way would be to employ the forthcoming first relation (76).

where the unit vectors

$$\hat{\mathbf{l}} = \begin{pmatrix} \cos\Omega \\ \sin\Omega \\ 0 \end{pmatrix} , \qquad \hat{\mathbf{m}} = \begin{pmatrix} -\cos\iota\sin\Omega \\ \cos\iota\cos\Omega \\ \sin\iota \end{pmatrix} , \tag{62}$$

point along the ascending node and perpendicular to it in the plane of the motion, respectively. Therefore once the reference plane and direction are chosen, $\hat{\mathbf{L}}_{\mathbf{N}}$ and $\hat{\mathbf{A}}_{\mathbf{N}}$ are equivalent with the angles $(\iota,\ \Omega)$ and ω, respectively.

A fourth orbital element is the time of periastron passage, t_0. Alternative options for the fourth orbital element are $\varepsilon = \tilde{\omega} + \mathcal{M}_0$ (the mean longitude at the epoch) and $\varepsilon^* = \varepsilon + n_N t - \int n_N dt$.

The parameter p and eccentricity of the orbit e are the two remaining orbital elements. From Eqs. (12). (22), and (23) we can express the dynamical constraints as function of them:

$$L_N^2 = Gm\mu^2 p \,, \tag{63}$$

$$A_N = Gm\mu e \,, \tag{64}$$

$$E_N = \frac{Gm\mu\left(e^2-1\right)}{2p} \,, \tag{65}$$

We see that for a given value of L_N^2 (fixing the parameter), the circular, elliptical, parabolic and elliptic orbits have the energies

$$\begin{aligned} E_N^{\text{circular}} &= -\frac{G^2m^2\mu^3}{2L_N^2} < 0 \,, \\ E_N^{\text{elliptic}} &= -\frac{G^2m^2\mu^3}{2L_N^2}\left(1-e^2\right) < 0 \,, \\ E_N^{\text{parabolic}} &= 0 \,, \\ E_N^{\text{hyperbolic}} &= \frac{G^2m^2\mu^3}{2L_N^2}\left(e^2-1\right) > 0 \,, \end{aligned} \tag{66}$$

respectively. Thus for a given L_N^2, the energies of the orbits obey $E_N^{\text{circular}} < E_N^{\text{elliptic}} < E_N^{\text{parabolic}} = 0 < E_N^{\text{hyperbolic}}$. In terms of dynamical constants the circular orbits are characterized by $A_N^{\text{circular}} = 0$, the parabolic orbits by $E_N^{\text{parabolic}} = 0$.

For elliptical and circular orbits the role of the parameter as an orbital element can be taken by the semimajor axis $a = p/\left(1-e^2\right)$. Eq. (65) then gives

$$E_N^{\text{circular, elliptic}} = -\frac{Gm\mu}{2a} \,. \tag{67}$$

2.6 Kinematical definition of the parametrizations for the elliptic and parabolic motions

The parametrization of the orbits in the plane of motion relies on the knowledge of their geometrical characteristics p and e. An alternative procedure, relying

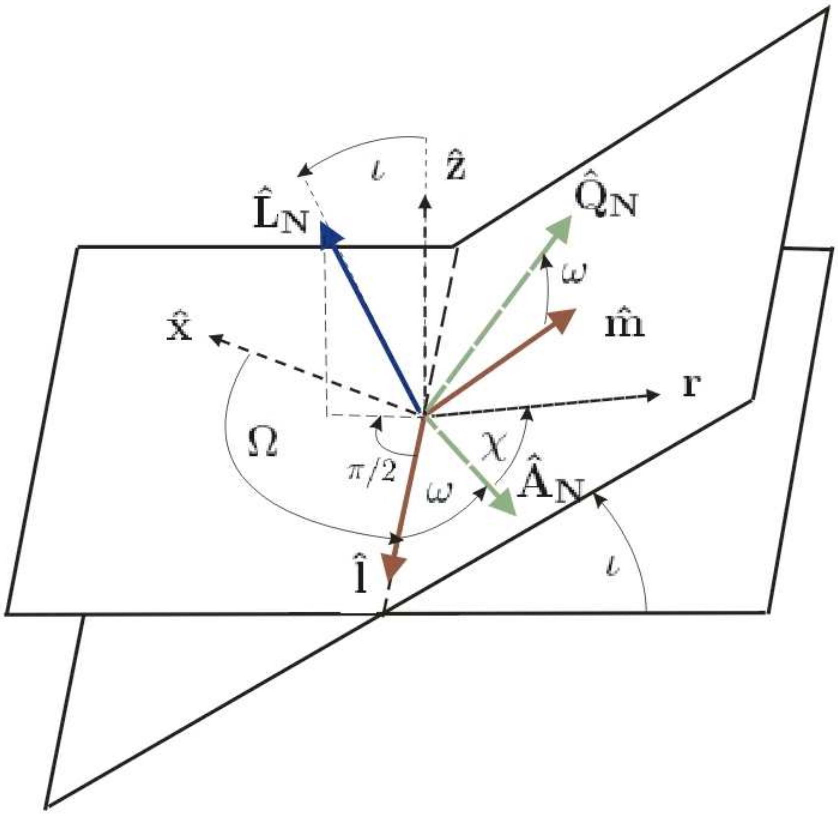

Figure 2: The horizontal plane, defined by its normal $\hat{\mathbf{z}}$ is the reference plane. On it, a reference direction $\hat{\mathbf{x}}$ is chosen. The plane tilted with the angle ι (the inclination) is the plane of motion, defined by the vectors $\mathbf{r}$, $\mathbf{v}$, or equivalently by its normal $\mathbf{L_N}$. On the plane of motion three orthonormal bases are frequently used. These are $\left\{\hat{\mathbf{r}},\ \hat{\mathbf{L}}_\mathbf{N}\times\hat{\mathbf{r}}\right\}$, $\left\{\hat{\mathbf{l}},\ \hat{\mathbf{m}}=\hat{\mathbf{L}}_\mathbf{N}\times\hat{\mathbf{l}}\right\}$ and $\left\{\hat{\mathbf{A}}_\mathbf{N},\ \hat{\mathbf{Q}}=\hat{\mathbf{L}}_\mathbf{N}\times\hat{\mathbf{A}}_\mathbf{N}\right\}$, where $\hat{\mathbf{l}}$ and $\hat{\mathbf{A}}_\mathbf{N}$ point along the ascending node (the intersection line of the two planes) and towards the periastron (the point of closest approach). The azimuthal angle for $\hat{\mathbf{l}}$ measured from $\hat{\mathbf{x}}$ in the reference plane is Ω (the longitude of the ascending node). The azimuthal angle for $\hat{\mathbf{A}}_\mathbf{N}$ measured from $\hat{\mathbf{l}}$ in the plane of motion is ω (the argument of the periastron). Note that the projection of $\hat{\mathbf{L}}_\mathbf{N}$ into the reference plane is also contained in the plane spanned by $\hat{\mathbf{L}}_\mathbf{N}$ and $\hat{\mathbf{z}}$. As both these vectors are perpendicular to $\hat{\mathbf{l}}$, so is the projection of $\hat{\mathbf{L}}_\mathbf{N}$. Therefore the azimuthal angle of $\hat{\mathbf{L}}_\mathbf{N}$ measured from $\hat{\mathbf{x}}$ in the reference plane is $\Omega-\pi/2$, while its polar angle is ι. The azimuthal and polar angles of the vector $\hat{\mathbf{m}}$ are $\pi/2+\Omega$ and $\pi/2-\iota$, respectively. Finally, the true anomaly χ is the azimuthal angle in the plane of motion of the position vector $\mathbf{r}$.

on kinematics, would be much useful for generalizing to a perturbed Kepler motion. Such a description can be built from the knowledge of the turning point(s), defined as $\dot{r} = 0$. This gives two turning points $r_{\substack{\max \\ \min}}$ for the ellipse and one turning point $r_{\min}$ for the parabola and hyperbola. Therefore a description based on the turning point(s) rather than (p, e) can be worked out for the elliptic and parabolic orbits (as the latter is characterized by p alone).

2.6.1 Elliptic orbits

For elliptic orbits an alternative way to find the solution (18) of the radial equation (16) is to follow the two steps:

(a.) determine the turning points $r_{\substack{\min \\ \max}}$ from the condition $\dot{r} = 0$. To Newtonian order the solution is $r_{\substack{\max \\ \min}} = r_{\pm}$ given by

$$r_{\pm} = \frac{L_N^2}{\mu(Gm\mu \mp A_N)} = \frac{Gm\mu \pm A_N}{-2E_N} \ , \tag{68}$$

(b1.) define the true anomaly parametrization as

$$\frac{2}{r} = \left(\frac{1}{r_{\min}} + \frac{1}{r_{\max}}\right) + \left(\frac{1}{r_{\min}} - \frac{1}{r_{\max}}\right) \cos\chi \ , \tag{69}$$

Then it is straightforward to verify that Eqs. (18) and (21) hold.

An equivalent form of the solution of the equations of motion can be given in terms of the eccentric anomaly ξ, an alternative definition of which is

(b2.)

$$2r = (r_{\max} + r_{\min}) - (r_{\max} - r_{\min}) \cos\xi \ , \tag{70}$$

It is immediate to verify that the following relation holding between the two parametrizations is equivalent with the definition of ξ given in the preceding subsection:

$$\tan\frac{\chi}{2} = \left(\frac{r_{\max}}{r_{\min}}\right)^{1/2} \tan\frac{\xi}{2} \ . \tag{71}$$

We also mention the follow-up relations:[4]

$$\begin{aligned} \sin\chi &= \frac{2\left(r_{\min} r_{\max}\right)^{1/2} \sin\xi}{(r_{\max} + r_{\min}) - (r_{\max} - r_{\min}) \cos\xi} \ , \\ \cos\chi &= -\frac{(r_{\max} - r_{\min}) - (r_{\max} + r_{\min}) \cos\xi}{(r_{\max} + r_{\min}) - (r_{\max} - r_{\min}) \cos\xi} \ . \end{aligned} \tag{74}$$

[4]These can be derived from the trigonometric relations

$$\cos\chi = \frac{1 - \tan^2\frac{\chi}{2}}{1 + \tan^2\frac{\chi}{2}} \ , \qquad \sin\chi = \frac{2\tan\frac{\chi}{2}}{1 + \tan^2\frac{\chi}{2}} \ , \tag{72}$$

and their inverse (written for ξ)

$$\tan^2\frac{\xi}{2} = \frac{1 - \cos\xi}{1 + \cos\xi} \ , \qquad \tan\frac{\xi}{2} = \frac{\sin\xi}{1 + \cos\xi} \ . \tag{73}$$

This form of the true and eccentric anomaly parametrizations will be particularly suitable in discussing perturbed Keplerian orbits.

Eqs. (74) can be rewritten by employing Eq. (70) as

$$
\begin{aligned}
r\sin\chi &= (r_{\min}r_{\max})^{1/2}\sin\xi \ , \\
2r\cos\chi &= (r_{\max}+r_{\min})\cos\xi-(r_{\max}-r_{\min}) \ .
\end{aligned} \tag{75}
$$

The turning points $r_{\substack{\max\\ \min}}=r_{\pm}$ being given by Eq. (68), we find $(r_{\min}r_{\max})^{1/2}=L_N/\left(-2\mu E_N\right)^{1/2}$, $r_{\max}+r_{\min}=Gm\mu/\left(-E_N\right)$ and $r_{\max}-r_{\min}=A_N/\left(-E_N\right)$. We obtain:

$$
\begin{aligned}
r\sin\chi &= \frac{L_N\sin\xi}{\left(-2\mu E_N\right)^{1/2}} \ , \\
r\cos\chi &= \frac{Gm\mu\cos\xi-A_N}{-2E_N} \ .
\end{aligned} \tag{76}
$$

2.6.2 Parabolic orbits

The true anomaly parametrization (69) can be extended for this case. In the limit $r_{\max}\rightarrow\infty$ it becomes

$$
r=\frac{2r_{\min}}{1+\cos\chi} \ , \tag{77}
$$

in accordance with the conic equation (24) and the earlier remark $p=2r_{\min}$ for the parabola. Therefore the true anomaly parametrization can be used in the description of parabolic orbits.

3 The perturbed two-body system

The dynamical constants and orbital elements of the Keplerian motion slowly vary due to perturbing forces. In this section we discuss this variation.

3.1 Lagrangian description of the perturbation

The motion of two bodies under the influence of generic perturbations can often be characterized by a Lagrangian:

$$\mathcal{L}\left(\mathbf{r},\mathbf{v},\mathbf{a},t\right)=\mathcal{L}_{N}\left(\mathbf{r},\mathbf{v}\right)+\Delta\mathcal{L}\left(\mathbf{r},\mathbf{v},\mathbf{a},t\right)\ , \tag{78}$$

where $\Delta\mathcal{L}$ represents the collection of perturbation terms. We allow $\Delta\mathcal{L}$ to be of second *differential* order (depending on coordinates, velocities and accelerations). The Euler-Lagrange equations in this case are:

$$\left(\frac{\partial}{\partial\mathbf{r}}-\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}+\frac{d^{2}}{dt^{2}}\frac{\partial}{\partial\mathbf{a}}\right)\mathcal{L}=0\ . \tag{79}$$

We define the acceleration in the perturbed case as

$$\mathbf{a}=\frac{1}{\mu}\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}\mathcal{L}_{N}\ \ . \tag{80}$$

Remembering the definition of the Newtonian acceleration, Eq. (2) and with the remark that the Newtonian part of the Lagrangian is acceleration-independent, we can rewrite Eq. (79) as:

$$\Delta\mathbf{a}\equiv\mathbf{a}-\mathbf{a}_{N}=\frac{1}{\mu}\left(\frac{\partial}{\partial\mathbf{r}}-\frac{d}{dt}\frac{\partial}{\partial\mathbf{v}}+\frac{d^{2}}{dt^{2}}\frac{\partial}{\partial\mathbf{a}}\right)\Delta\mathcal{L}\ . \tag{81}$$

The quantity $\Delta\mathbf{a}$ is of the order of perturbations. Because of this, whenever accelerations or its derivatives appear when the right hand side of the above equation is evaluated, they can be replaced by the corresponding Newtonian expressions.

3.2 Evolution of the Keplerian dynamical constants in terms of the perturbing force

We decompose the sum of the perturbing forces per unit mass $\Delta\mathbf{a}$ in the (slowly evolving) basis $\{\mathbf{f_i}\}$ as:

$$\Delta\mathbf{a}=\alpha\hat{\mathbf{A}}_{\mathbf{N}}+\beta\hat{\mathbf{Q}}_{\mathbf{N}}+\gamma\hat{\mathbf{L}}_{\mathbf{N}}\ . \tag{82}$$

We can apply the forthcoming description of the perturbed Keplerian motion based on the components α, β, γ even when a Lagrangian description is not available.

The Keplerian dynamical constants are not related to the symmetries of the perturbed dynamics, and are not constants of motion in general. Nevertheless, they are useful in monitoring the evolution of the orbital elements, as will be discussed later in this Section. In this subsection we wish to determine their variation as function of the components $\alpha,\ \beta,\ \gamma$ of the perturbing force.

The time derivative of Eq. (14) gives

$$\dot{E}_N = \mu \mathbf{v} \cdot \Delta \mathbf{a} \ . \tag{83}$$

In the basis $\left\{ \mathbf{f_{(i)}} \right\} = (\hat{\mathbf{A}}_\mathbf{N},\ \hat{\mathbf{Q}}_\mathbf{N},\ \hat{\mathbf{L}}_\mathbf{N})$ it becomes:

$$\dot{E}_N = \mu \left(\tau\alpha + \lambda\beta \right) \ , \tag{84}$$

with the coefficients given in Eqs. (35). We note that the angle χ involved in these coefficients is spanned by $\hat{\mathbf{A}}_\mathbf{N}$ and $\hat{\mathbf{r}}$, as in the unperturbed case and will be denoted χ_p in what follows.

The Newtonian orbital angular momentum evolves as:

$$\dot{\mathbf{L}}_\mathbf{N} = \mu \mathbf{r} \times \Delta \mathbf{a} \ . \tag{85}$$

In the chosen basis we find

$$\dot{\mathbf{L}}_\mathbf{N} = \gamma \left(\sigma \hat{\mathbf{A}}_\mathbf{N} - \rho \hat{\mathbf{Q}}_\mathbf{N} \right) + \left(\rho\beta - \sigma\alpha \right) \hat{\mathbf{L}}_\mathbf{N} \ . \tag{86}$$

Employing the generic formula for the time derivative of any vector $\mathbf{V}$,

$$\dot{\mathbf{V}} = \dot{V} \hat{\mathbf{V}} + V \frac{d}{dt} \hat{\mathbf{V}} \ , \tag{87}$$

both the evolution of the magnitude and of the direction of the Newtonian orbital angular momentum are found:

$$\dot{L}_N = \rho\beta - \sigma\alpha \ , \tag{88}$$

$$\frac{d}{dt} \hat{\mathbf{L}}_\mathbf{N} = \frac{\gamma}{L_N} \left(\rho \hat{\mathbf{A}}_\mathbf{N} + \sigma \hat{\mathbf{Q}}_\mathbf{N} \right) \times \hat{\mathbf{L}}_\mathbf{N} = \gamma \frac{\mu}{L_N} \mathbf{r} \times \hat{\mathbf{L}}_\mathbf{N} \ , \tag{89}$$

The last equation describes the change of the orbital plane.

The Laplace-Runge-Lenz vector also evolves in the presence of perturbations. This can be seen by computing the time derivatives of Eq. (11) and recalling that $\mathbf{A}_\mathbf{N}$ is a constant of the Keplerian motion:

$$\begin{aligned} \dot{\mathbf{A}}_\mathbf{N} &= \Delta \mathbf{a} \times \mathbf{L}_\mathbf{N} + \mathbf{v} \times \dot{\mathbf{L}}_\mathbf{N} \\ &= \mu \left[2 \left(\mathbf{v} \cdot \Delta \mathbf{a} \right) \mathbf{r} - \left(\mathbf{r} \cdot \Delta \mathbf{a} \right) \mathbf{v} - \left(\mathbf{r} \cdot \mathbf{v} \right) \Delta \mathbf{a} \right] \ . \end{aligned} \tag{90}$$

Computation gives

$$\dot{\mathbf{A}}_\mathbf{N} = \left(\beta L_N + \lambda \dot{L}_N \right) \hat{\mathbf{A}}_\mathbf{N} - \left(\alpha L_N + \tau \dot{L}_N \right) \hat{\mathbf{Q}}_\mathbf{N} - \gamma \left(\tau\rho + \lambda\sigma \right) \hat{\mathbf{L}}_\mathbf{N} \ , \tag{91}$$

from which we identify

$$\dot{A}_N = \beta L_N + \lambda \dot{L}_N \ , \tag{92}$$

$$\frac{d}{dt} \hat{\mathbf{A}}_\mathbf{N} = \frac{1}{A_N} \left[\gamma \left(\tau\rho + \lambda\sigma \right) \hat{\mathbf{Q}}_\mathbf{N} - \left(\alpha L_N + \tau \dot{L}_N \right) \hat{\mathbf{L}}_\mathbf{N} \right] \times \hat{\mathbf{A}}_\mathbf{N} \ , \tag{93}$$

The last equation describes the evolution of the periastron, composed from a precessional part in the plane of the motion (about $\hat{\mathbf{L}}_{\mathbf{N}}$) and a coevolution with the plane of motion.

With the expression of the coefficients (35) the evolution of E_N, L_N and A_N are obtained explicitly as:

$$\begin{aligned}
\dot{E}_N &= -\alpha\frac{Gm\mu^2}{L_N}\sin\chi_p + \beta\frac{\mu\left(A_N + Gm\mu\cos\chi_p\right)}{L_N} \ , \\
\dot{L}_N &= \mu r\left(\beta\cos\chi_p - \alpha\sin\chi_p\right) \ , \\
\dot{A}_N &= \beta L_N + \frac{A_N + Gm\mu\cos\chi_p}{L_N}\mu r\left(\beta\cos\chi_p - \alpha\sin\chi_p\right) \ . \qquad (94)
\end{aligned}$$

Note that the time derivative of Eq. (12) gives an identity with the above evolutions and employing the true anomaly parametrization (18), therefore the algebraic relation (12) continue to hold in the perturbed case.

The evolution of $\hat{\mathbf{A}}_{\mathbf{N}}$ and $\hat{\mathbf{L}}_{\mathbf{N}}$ will be given in explicit form in the next subsection.

3.3 Precession of the basis vectors in terms of the perturbing force

As a by-product of the calculations of the subsection 3.2, we also obtained the precession of two of the basis vectors $\left\{\mathbf{f}_{(\mathbf{i})}\right\}$. The precession of the remaining basis vector of the basis $\left\{\mathbf{f}_{(\mathbf{i})}\right\}$ can be derived from its definition $\hat{\mathbf{Q}}_{\mathbf{N}} = \hat{\mathbf{L}}_{\mathbf{N}} \times \hat{\mathbf{A}}_{\mathbf{N}}$.

$$\frac{d}{dt}\hat{\mathbf{Q}}_{\mathbf{N}} = \left[\gamma\frac{\rho}{L_N}\hat{\mathbf{A}}_{\mathbf{N}} - \frac{\alpha L_N + \tau\dot{L}_N}{A_N}\hat{\mathbf{L}}_{\mathbf{N}}\right] \times \hat{\mathbf{Q}}_{\mathbf{N}} \ . \qquad (95)$$

Inserting the explicit expression (35) for the coefficients $\rho, \sigma, \tau, \lambda$ (with χ_p in place of χ) in the Eqs. (89), (93) and (95) we obtain the explicit expressions of the precessional motion of the basis vectors (31):

$$\dot{\mathbf{f}}_{(\mathbf{i})} = \mathbf{\Omega}_{(\mathbf{i})} \times \mathbf{f}_{(\mathbf{i})} \ , \qquad (96)$$

with

$$\begin{aligned}
\mathbf{\Omega}_{(\mathbf{1})} &= \gamma\frac{\mu r\sin\chi_p}{L_N}\hat{\mathbf{Q}}_{\mathbf{N}} \\
&\quad - \left[\alpha\frac{L_N}{A_N} + \left(\alpha\sin\chi_p - \beta\cos\chi_p\right)\frac{Gm\mu^2 r\sin\chi_p}{L_N\ A_N}\right]\hat{\mathbf{L}}_{\mathbf{N}} \ , \qquad (97) \\
\mathbf{\Omega}_{(\mathbf{2})} &= \gamma\frac{\mu r\cos\chi_p}{L_N}\hat{\mathbf{A}}_{\mathbf{N}} \\
&\quad - \left[\alpha\frac{L_N}{A_N} + \left(\alpha\sin\chi_p - \beta\cos\chi_p\right)\frac{Gm\mu^2 r\sin\chi_p}{L_N\ A_N}\right]\hat{\mathbf{L}}_{\mathbf{N}} \ , \qquad (98) \\
\mathbf{\Omega}_{(\mathbf{3})} &= \gamma\frac{\mu r\cos\chi_p}{L_N}\hat{\mathbf{A}}_{\mathbf{N}} + \gamma\frac{\mu r\sin\chi_p}{L_N}\hat{\mathbf{Q}}_{\mathbf{N}} \ , \qquad (99)
\end{aligned}$$

As $\mathbf{f_{(i)}} \times \mathbf{f_{(i)}} = 0$, we can add to the above expressions of $\Omega_{(\mathbf{i})}$ terms proportional to $\mathbf{f_{(i)}}$, such that we have a single angular velocity vector Ω, with components in the $\{\mathbf{f_{(i)}}\}$ basis:

$$\mathbf{\Omega}_j = \left(\gamma \frac{\mu r \cos\chi_p}{L_N}, \gamma \frac{\mu r \sin\chi_p}{L_N}, -\left[\alpha \frac{L_N}{A_N} + (\alpha \sin\chi_p - \beta \cos\chi_p) \frac{Gm\mu^2 r \sin\chi_p}{L_N A_N}\right]\right) . \tag{100}$$

Then

$$\dot{\mathbf{f}}_{(\mathbf{i})} = \mathbf{\Omega} \times \mathbf{f_{(i)}} . \tag{101}$$

The expressions (83)-(100) manifestly vanish in the Newtonian limit together with either $\Delta\mathbf{a}$ or the coefficients α, β and γ.

A couple of immediate remarks are in order:

(a) If $\gamma = 0$ (no perturbing force is pointing outside the plane of motion), $\hat{\mathbf{L}}_\mathbf{N}$ (the plane of motion) is conserved, while both $\hat{\mathbf{A}}_\mathbf{N}$ and $\hat{\mathbf{Q}}_\mathbf{N}$ undergo a precessional motion about $\hat{\mathbf{L}}_\mathbf{N}$ (in the conserved plane of motion).

(b) If $\alpha = \beta = 0$ (the perturbing force is perpendicular to the plane of motion), then $\hat{\mathbf{A}}_\mathbf{N}$ undergoes a precessional motion about $\hat{\mathbf{Q}}_\mathbf{N}$ and vice-versa, while $\hat{\mathbf{L}}_\mathbf{N}$ about $\mathbf{r}$.

3.4 Summary

The perturbed bounded Keplerian motion can be visualized as a Keplerian ellipse with slowly varying parameters. The semimajor axis a and eccentricity e can be defined exactly as in the unperturbed case, from the dynamical quantities E_N and A_N. The evolution of $a,\ e$ can be inferred then from the evolutions of $E_N,\ A_N$. The orbit therefore rather of being an exact ellipse should be thought of as a curve which can be approximated in each of its points by an ellipse with these slowly varying parameters $a,\ e$. This description is known as an osculating ellipse.

Whenever the force depends on χ_p only, the evolution equations (94) (supplemented with the additional evolution equations of any dynamical variable - like the spins - possibly contained in $\alpha,\ \beta$) give the *radial evolution* of the binary system (provided an equation $dt/d\chi_p$ holding in the perturbed case is provided - this will be given in chapter 5.1). The additional equations (89) and (93) give the *angular evolutions*, e.g. the evolution of the orbital plane and of the precession of the periastron in the orbital plane.

The equations (84), (89), (92) and (93) are equivalent with the Lagrange planetary equations for the orbital elements $(a, e, \iota, \Omega, \omega)$ of the osculating orbit, as will be shown in the next section. However the use of the above mentioned system of equations has the undoubted advantage of being independent of the choice of the reference plane and axis.

4 The Lagrange planetary equations

In chapter 3 we have discussed in detail the evolution of a perturbed Keplerian system, relying on the corresponding evolution of the Keplerian dynamical constants. An alternative approach to this problem is represented by the Lagrange planetary equations. In this section we prove that they are equivalent to the evolution equations of the dynamical constants.

4.1 Evolution of the orbital elements in terms of the evolution of Keplerian dynamical constants

We have seen that in the presence of perturbations the Keplerian dynamical constants E_N, $\mathbf{L_N}$ and $\mathbf{A_N}$ slowly vary. Due to perturbations a related slow evolution of the orbital elements also occurs. In this subsection we determine the latter as function of the former.

From Eqs. (63) and (64) we find

$$\begin{aligned} \dot{\mathbf{L}}_{\mathbf{N}} &= \left[Gm\mu^2 a(1-e^2)\right]^{1/2} \\ &\quad\times \left[\left(\frac{\dot{a}}{2a} - \frac{e\dot{e}}{1-e^2}\right)\hat{\mathbf{L}}_{\mathbf{N}} + \frac{d}{dt}\hat{\mathbf{L}}_{\mathbf{N}}\right] , \quad (102) \\ \dot{\mathbf{A}}_{\mathbf{N}} &= Gm\mu\left(\dot{e}\hat{\mathbf{A}}_{\mathbf{N}} + e\frac{d}{dt}\hat{\mathbf{A}}_{\mathbf{N}}\right) \quad (103) \end{aligned}$$

The time-derivatives of Eqs. (61)-(62) give generic expressions for the time-derivatives of $\hat{\mathbf{l}}$, $\hat{\mathbf{m}}$ and $\hat{\mathbf{L}}_{\mathbf{N}}$ in terms of the time-derivatives of the angular orbital elements:

$$\begin{aligned} \frac{d}{dt}\hat{\mathbf{l}} &= \dot{\Omega}\left(\cos\iota\ \hat{\mathbf{m}} - \sin\iota\ \hat{\mathbf{L}}_{\mathbf{N}}\right) , \quad (104) \\ \frac{d}{dt}\hat{\mathbf{m}} &= \dot{\iota}\ \hat{\mathbf{L}}_{\mathbf{N}} - \dot{\Omega}\cos\iota\ \hat{\mathbf{l}} , \quad (105) \\ \frac{d}{dt}\hat{\mathbf{L}}_{\mathbf{N}} &= -\dot{\iota}\ \hat{\mathbf{m}} + \dot{\Omega}\sin\iota\ \hat{\mathbf{l}} , \quad (106) \end{aligned}$$

From these and the time derivatives of Eqs. (60) we obtain:

$$\begin{aligned} \frac{d}{dt}\hat{\mathbf{A}}_{\mathbf{N}} &= \left(\dot{\omega} + \dot{\Omega}\cos\iota\right)\ \hat{\mathbf{Q}}_{\mathbf{N}} \\ &\quad + \left(\dot{\iota}\sin\omega - \dot{\Omega}\sin\iota\cos\omega\right)\ \hat{\mathbf{L}}_{\mathbf{N}} , \quad (107) \\ \frac{d}{dt}\hat{\mathbf{Q}}_{\mathbf{N}} &= -\left(\dot{\omega} + \dot{\Omega}\cos\iota\right)\hat{\mathbf{A}}_{\mathbf{N}} \\ &\quad + \left(\dot{\iota}\cos\omega + \dot{\Omega}\sin\iota\sin\omega\right)\ \hat{\mathbf{L}}_{\mathbf{N}} . \quad (108) \end{aligned}$$

Then the projections of the Eqs. (102)-(103) give the time derivatives of 5 orbital elements as functions of the time derivatives of the Keplerian dynamical constants:

$$
\begin{aligned}
\dot{\iota} &= -\frac{\dot{\mathbf{L}}_{\mathbf{N}} \cdot \hat{\mathbf{m}}}{\left[Gm\mu^2 a(1-e^2)\right]^{1/2}} \ , \qquad (109)\\
\dot{\Omega} &= \frac{\dot{\mathbf{L}}_{\mathbf{N}} \cdot \hat{\mathbf{l}}}{\left[Gm\mu^2 a(1-e^2)\right]^{1/2}\sin\iota} \ , \qquad (110)\\
\dot{\omega} &= \frac{\dot{\mathbf{A}}_{\mathbf{N}} \cdot \hat{\mathbf{Q}}_{\mathbf{N}}}{Gm\mu e} - \frac{\left(\dot{\mathbf{L}}_{\mathbf{N}} \cdot \hat{\mathbf{l}}\right)\cot\iota}{\left[Gm\mu^2 a(1-e^2)\right]^{1/2}} \ , \qquad (111)\\
\dot{a} &= 2a\left[\frac{\dot{\mathbf{L}}_{\mathbf{N}} \cdot \hat{\mathbf{L}}_{\mathbf{N}}}{\left[Gm\mu^2 a(1-e^2)\right]^{1/2}} + \frac{e\left(\dot{\mathbf{A}}_{\mathbf{N}} \cdot \hat{\mathbf{A}}_{\mathbf{N}}\right)}{Gm\mu\left(1-e^2\right)}\right] \ , \qquad (112)\\
\dot{e} &= \frac{\dot{\mathbf{A}}_{\mathbf{N}} \cdot \hat{\mathbf{A}}_{\mathbf{N}}}{Gm\mu} \ . \qquad (113)
\end{aligned}
$$

4.2 Evolution of the orbital elements in terms of the perturbing force

Based on the expressions

$$
\begin{aligned}
\dot{\mathbf{L}}_{\mathbf{N}} &= L_N\Omega_2\hat{\mathbf{A}}_{\mathbf{N}} - L_N\Omega_1\hat{\mathbf{Q}}_{\mathbf{N}} + \dot{L}_N\hat{\mathbf{L}}_{\mathbf{N}} \ , \qquad (114)\\
\dot{\mathbf{A}}_{\mathbf{N}} &= \dot{A}_N\hat{\mathbf{A}}_{\mathbf{N}} + A_N\Omega_3\hat{\mathbf{Q}}_{\mathbf{N}} - A_N\Omega_2\hat{\mathbf{L}}_{\mathbf{N}} \ , \qquad (115)
\end{aligned}
$$

with the coefficients derived in the previous section, the relations (60) between the basis vectors in the plane of motion, and on Eqs. (63) and (64) giving L_N and A_N in terms of orbital elements, the variation (109)-(113) of the orbital elements can be expressed in terms of the perturbing force components as:

$$
\begin{aligned}
\dot{\iota} &= \Omega_1\cos\omega - \Omega_2\sin\omega \ , \qquad (116)\\
\dot{\Omega} &= \frac{\Omega_2\cos\omega + \Omega_1\sin\omega}{\sin\iota} \ , \qquad (117)\\
\dot{\omega} &= \Omega_3 - \left(\Omega_2\cos\omega + \Omega_1\sin\omega\right)\cot\iota \ , \qquad (118)\\
\dot{a} &= 2a\left[\frac{\dot{L}_N}{\left[Gm\mu^2 a(1-e^2)\right]^{1/2}} + \frac{e\dot{A}_N}{Gm\mu\left(1-e^2\right)}\right] \ , \qquad (119)\\
\dot{e} &= \frac{\dot{A}_N}{Gm\mu} \ . \qquad (120)
\end{aligned}
$$

From the above relations and the expression of the angular velocities (100) one readily finds that

$$
\dot{\Omega} = \dot{\iota}\frac{\tan\left(\omega + \chi_p\right)}{\sin\iota} \ . \qquad (121)
$$

The expressions giving the variation of the angular orbital elements can be given explicitly by inserting the corresponding elements of the angular velocities (100) and employing Eq. (18) given in terms of the orbital elements as

$$r=\frac{a\left(1-e^{2}\right)}{1+e\cos\chi_{p}}\ , \tag{122}$$

together with Eq. (63). We obtain:

$$\dot{\iota} = \gamma\left[\frac{a\left(1-e^{2}\right)}{Gm}\right]^{1/2}\frac{\cos\left(\omega+\chi_{p}\right)}{1+e\cos\chi_{p}}\ , \tag{123}$$

$$\dot{\Omega} = \gamma\left[\frac{a\left(1-e^{2}\right)}{Gm}\right]^{1/2}\frac{\sin\left(\omega+\chi_{p}\right)}{\sin\iota\ \left(1+e\cos\chi_{p}\right)}\ , \tag{124}$$

$$\dot{\omega} = -\left[\frac{a\left(1-e^{2}\right)}{Gm}\right]^{1/2} \times\frac{\alpha\left(1+e\cos\chi_{p}+\sin^{2}\chi_{p}\right)-\beta\sin\chi_{p}\cos\chi_{p}+\gamma e\cot\iota\sin\left(\omega+\chi_{p}\right)}{e\left(1+e\cos\chi_{p}\right)} \tag{125}$$

Thus the angular orbital elements ι and Ω are changed only by perturbing forces perpendicular to the plane of motion, while ω is changed by any perturbing force. If the perturbing force is such that the plane of motion is unchanged ($\gamma=0$), then $\hat{\mathbf{l}}$ is also unaffected, and the change in ω can be interpreted as pure periastron precession: $\dot{\omega}=\Omega_{3}$ and $\Delta\omega=\Delta\psi_{0}$. (Otherwise stated $\gamma=0$ implies $\Omega_{1}=\Omega_{2}=0$.)

Note that the α, β-parts of $\dot{\omega}$ become ill-defined in the limit $e\rightarrow0$. This is however not surprising, as ω cannot be defined for circular orbits.

The expressions (119)-(120) can be further expanded by employing Eqs. (94), obtaining

$$\dot{a} = \frac{2a^{3/2}}{\left[Gm\left(1-e^{2}\right)\right]^{1/2}}\left[\beta\left(e+\cos\chi_{p}\right)-\alpha\sin\chi_{p}\right]\ , \tag{126}$$

$$\dot{e} = \left[\frac{a(1-e^{2})}{Gm}\right]^{1/2}\frac{\beta\left(1+2e\cos\chi_{p}+\cos^{2}\chi_{p}\right)-\alpha\left(e+\cos\chi_{p}\right)\sin\chi_{p}}{1+e\cos\chi_{p}} \tag{127}$$

Thus the orbital elements a, e are not changed by perturbing forces perpendicular to the plane of motion.

Eqs. (123)-(125) and (126)-(127) are the Lagrange planetary equations.

5 Orbital evolution under perturbations in terms of the true anomaly

In this section we return to the characterization of the orbit and dynamics in terms of the dynamical constants of the Keplerian motion. Although the Lagrange planetary equations are widely used, we find more convenient to discuss the perturbed Keplerian problem based on the equivalent set of equations (84), (89), (92) and (93), derived earlier in the subsection 3.2.

5.1 The perturbed Newtonian dynamical constants

As the basis$\{\mathbf{f_{(i)}}\}$ is comoving with the plane of motion and the periastron, the position vector

$$\mathbf{r} = x^i \mathbf{f_{(i)}} \tag{128}$$

with components

$$x^1 = r\cos\chi_p , \qquad x^2 = r\sin\chi_p , \qquad x^3 = 0 \tag{129}$$

changes according to (see also Appendix A)

$$\begin{aligned} \mathbf{v} &= \dot{x}^i \mathbf{f_{(i)}} + x^i \dot{\mathbf{f}}_{\mathbf{(i)}} = \dot{x}^i \mathbf{f_{(i)}} + x^i \mathbf{\Omega} \times \mathbf{f_{(i)}} \\ &= \left(\dot{x}^i - \epsilon^i{}_j{}^k x^j \Omega_k\right) \mathbf{f_{(i)}} \ , \end{aligned} \tag{130}$$

where $\epsilon^i{}_j{}^k$ is the antisymmetric Levi-Civita symbol and $\dot{x}^i$ are found as the time derivatives of the coordinates (129).

Then, starting from the definition of the Newtonian orbital angular momentum, Eq. (61), and employing the relations (128)-(130) and the angular velocities (100), we obtain by straightforward algebra (see Appendix A)

$$\mathbf{L_N} = \mu r^2 \left(\dot{\chi}_p + \Omega_3\right) \hat{\mathbf{L}}_{\mathbf{N}} \ . \tag{131}$$

This reduces to the manifestly Newtonian expression, whenever $\Omega_3 = 0$, thus whenever $\hat{\mathbf{A}}_{\mathbf{N}}$ does not precess about $\hat{\mathbf{L}}_{\mathbf{N}}$.

Similarly, starting from the definition of the energy, Eq. (9), by the same method (see Appendix A) we obtain

$$E_N = \frac{\mu\left(\dot{r}^2 + r^2\dot{\chi}_p^2\right)}{2} - \frac{Gm\mu}{r} + \mu r^2 \dot{\chi}_p \ \Omega_3 + \frac{\mu r^2}{2}\Omega_3^2 \ . \tag{132}$$

(The last $\mathcal{O}\left(\alpha^2 ,\ \alpha\beta ,\ \beta^2\right)$ term can be dropped in a linear approximation.) Again, whenever $\Omega_3 = 0$, the Newtonian expression is recovered.

Therefore the Newtonian energy and Newtonian angular momentum share the properties, that modulo Ω_3 terms they reduce to manifestly Newtonian forms. We can express $\dot{\chi}_p$ from Eq. (131) as[5]

$$\dot{\chi}_p = \frac{L_N}{\mu r^2} - \Omega_3 \ . \tag{134}$$

Inserting this to Eq. (132), we obtain the radial equation

$$\dot{r}^2 = \frac{2E_N}{\mu} + \frac{2Gm}{r} - \frac{L_N^2}{\mu^2 r^2} \ . \tag{135}$$

Finally, starting from the definition of the Laplace-Runge-Lenz vector, Eq. (11) and employing the relations (128)-(130) and (131), we can deduce (see Appendix A) its evolution as

$$\begin{aligned} \mathbf{A_N} = & \left[\mu r^2 \dot{\chi}_p \left(\dot{r}\sin\chi_p + r\dot{\chi}_p \cos\chi_p\right) - Gm\mu\cos\chi_p \right. \\ & \left. + \mu r^2 \Omega_3 \left(\dot{r}\sin\chi_p + 2r\dot{\chi}_p\cos\chi_p + r\Omega_3\cos\chi_p\right)\right] \mathbf{\hat{A}_N} \\ & - \left[\mu r^2 \dot{\chi}_p \left(\dot{r}\cos\chi_p - r\dot{\chi}_p\sin\chi_p\right) + Gm\mu\sin\chi_p \right. \\ & \left. + \mu r^2\Omega_3 \left(\dot{r}\cos\chi_p - 2r\dot{\chi}_p\sin\chi_p - \Omega_3 r\sin\chi_p\right)\right] \mathbf{\hat{Q}_N} \ . \end{aligned} \tag{136}$$

By inserting in the above relation $\dot{\chi}_p$ given by Eq. (134), all Ω_3 terms cancel out and we obtain:

$$\begin{aligned} \mathbf{A_N} = & \left[\left(\frac{L_N^2}{\mu r} - Gm\mu\right)\cos\chi_p + L_N\dot{r}\sin\chi_p\right] \mathbf{\hat{A}_N} \\ & + \left[\left(\frac{L_N^2}{\mu r} - Gm\mu\right)\sin\chi_p - L_N\dot{r}\cos\chi_p\right] \mathbf{\hat{Q}_N} \ . \end{aligned} \tag{137}$$

As $\mathbf{A_N} = A_N \mathbf{\hat{A}_N}$:

$$A_N = \left(\frac{L_N^2}{\mu r} - Gm\mu\right)\cos\chi_p + L_N\dot{r}\sin\chi_p \ , \tag{138}$$

$$0 = \left(\frac{L_N^2}{\mu r} - Gm\mu\right)\sin\chi_p - L_N\dot{r}\cos\chi_p \ . \tag{139}$$

Solving for r and $\dot{r}$ gives

$$\begin{aligned} \frac{L_N^2}{\mu r} - Gm\mu &= A_N \cos\chi_p \ , \\ L_N \dot{r} &= A_N \sin\chi_p \ . \end{aligned} \tag{140}$$

The derived expressions $r\left(\chi_p\right)$ and $\dot{r}\left(\chi_p\right)$ are identical with the zeroth order true anomaly parametrization $r\left(\chi\right)$ and zeroth order expression of $\dot{r}\left(\chi\right)$, Eqs.

[5] This also arises from the relation

$$\dot{\omega} + \dot{\chi}_p = \frac{L_N}{\mu r^2} - \dot{\Omega}\cos\iota \tag{133}$$

derived in Ref. [1], after inserting the relations (117) and (118).

(18) and (20), respectively. Therefore the true anomaly parametrization χ_p is introduced exactly in the same way than in the unperturbed case.

As a consistency check, we calculate the derivative of $r(\chi_p)$ as

$$\dot{r} = \frac{\mu A_N r^2 \dot{\chi}_p \sin\chi_p}{L_N^2} + \frac{2r\dot{L}_N}{L_N} - \frac{\mu r^2 \dot{A}_N \cos\chi_p}{L_N^2} \ , \tag{141}$$

and put it equal to $\dot{r}(\chi_p)$, derived earlier. We insert $\dot{\chi}_p$ as given by Eq. (134); also $\dot{L}_N$ and $\dot{A}_N$ as given by the Eqs (94). As a result, we re-obtain the expression of Ω_3 given in Eq. (100).

A second consistency check involving $r(\chi_p)$, $\dot{r}(\chi_p)$ and the radial equation (135) also holds, as in the Keplerian motion.

5.2 The evolution of the Newtonian dynamical constants in terms of the true anomaly

The scalars $E_N,\ L_N,\ A_N$ become χ_p-dependent in the presence of a perturbing force. We can decompose these Newtonian expressions in the unperturbed (Keplerian) part, and a χ_p-dependent correction as $E_N = E_N^0 + E_N^1(\chi_p)$, $L_N = L_N^0 + L_N^1(\chi_p)$, $A_N = A_N^0 + A_N^1(\chi_p)$. Their explicit expression can be derived starting from their evolution equations (94), by passing from time-derivatives to derivatives with respect to χ_p. As all terms in the equations are first order, we simply employ $\dot{\chi}_p = L_N^0/\mu r^2$ and obtain:

$$\begin{aligned}
\frac{dE_N^1}{d\chi_p} &= \frac{Gm\mu^3}{\left(L_N^0\right)^2} r^2 \left[-\alpha\sin\chi_p + \beta\left(\frac{A_N^0}{Gm\mu} + \cos\chi_p\right)\right] \ , \\
\frac{dL_N^1}{d\chi_p} &= \frac{\mu^2}{L_N^0} r^3 \left(\beta\cos\chi_p - \alpha\sin\chi_p\right) \ , \\
\frac{dA_N^1}{d\chi_p} &= \mu\beta r^2 + \frac{Gm\mu^3}{\left(L_N^0\right)^2} r^3 \left(\beta\cos\chi_p - \alpha\sin\chi_p\right)\left(\frac{A_N^0}{Gm\mu} + \cos\chi_p\right) \ .
\end{aligned} \tag{142}$$

By inserting $r(\chi_p) = \left(L_N^0\right)^2/\mu\left(Gm\mu + A_N^0\cos\chi_p\right)$ and provided the components of the perturbing force can be given as functions of χ_p alone (that is, they can possibly depend on r and χ_p only), Eqs. (142) become ordinary differential equations for $E_N^1,\ L_N^1$ and A_N^1. Unless $\alpha, \beta \propto r^{n\leq -3}$ however, the integrals may be cumbersome to evaluate due to the negative powers of $\left(Gm\mu + A_N^0\cos\chi_p\right)$ in the integrands. In these cases the use of the eccentric anomaly parametrization may simplify the calculations (similarly as in the derivation of the unperturbed Kepler equation). Therefore in the next section we will work out the details of this parametrization applying for the perturbed case.

5.3 Summary of the results obtained with the true anomaly parametrization

The forces perturbing the Keplerian orbit do not change the true anomaly parametrization $r(\chi_p)$ and $\dot{r}(\chi_p)$, Eqs. Eq. (18) and (20), respectively, with χ_p

in place of χ. In the order of accuracy of the perturbations the radial equation (16) also holds. These in turn mean that the eccentric anomaly parametrization ξ_p can be introduced exactly as in the Keplerian case. We can also define a radial period as the time in which the mass μ moves from an $r_{\min}$ location given by $\dot{r} = 0$ to the following $r_{\min}$, given by the second forthcoming $\dot{r} = 0$ locus (the first will give $r_{\max}$).

The following, however are changed with respect to the Keplerian case:

(a) The basis $\left\{\mathbf{f_{(i)}}\right\}$ constructed from the Keplerian dynamical constants slowly evolves, with precessional angular velocities Ω_j given as linear and homogeneous expressions of the components of the resulting perturbing force.

(b) The plane of motion, determined by $\mathbf{\hat{L}_N}$, can change due to perturbation forces transverse to the plane of motion.

(c) The periastron, determined by $\mathbf{\hat{A}_N}$, undergoes a composed evolution. It has a precession in the plane of motion due to the forces acting in the plane of motion (this is characterized by Ω_3, the precessional angular velocity of $\mathbf{\hat{A}_N}$ about $\mathbf{\hat{L}_N}$) and it also moves with the plane of motion.

(d) The precession of the periastron means that the orbit fails to be closed. The χ_p-dependence of the Ω_3 angular velocity component signifies that at each moment the movement of the periastron of the corresponding osculating ellipse will be different. The change of $\mathbf{\hat{A}}_N$ over one radial period will give the periastron shift.

(e) When expressed in terms of the true anomaly χ_p, both the Newtonian orbital angular momentum vector and the Newtonian energy acquire linear terms in Ω_3, given in Eqs. (131) and (132), respectively.

(f) The scalars $E_N,\ L_N,\ A_N$ evolve in χ_p (or equivalently, in ξ_p).

(g) The expression $\dot{\chi}_p\left(r\right)$ is changed as compared to the corresponding Keplerian expression by a linear term in Ω_3 [Eq. (134)].

6 Orbital evolution under perturbations in terms of the eccentric anomaly

In this section, besides rewriting all results of the previous section in terms of the eccentric anomaly, we derive the perturbed Kepler equation.

6.1 The eccentric anomaly parametrization

We introduce the eccentric anomaly parametrization similarly as in the Keplerian case [by Eq. (54), with ξ_p in place of ξ]:

$$r = \frac{Gm\mu - A_N \cos\xi_p}{-2E_N} \ . \tag{143}$$

From the expressions of $\dot{r}\,(\chi_p)$ [given by the second relation (140) and employing Eq. (76)], and of $\dot{\chi}_p$ [given by Eqs. (134) and the third component of the expression (100)] we obtain $\dot{r}$ and $\dot{\chi}_p$ in terms of the eccentric anomaly:

$$\dot{r} = \frac{A_N}{\left(-2\mu E_N\right)^{1/2} r} \sin\xi_p \ , \tag{144}$$

$$\begin{aligned} r\dot{\chi}_p &= \frac{L_N}{\mu r} + \alpha \frac{L_N}{A_N}\left(r + \frac{Gm\mu}{-2E_N}\sin^2\xi_p\right) \\ &\quad - \beta\left(\frac{\mu}{-2E_N}\right)^{1/2} \frac{Gm\mu}{-2E_N A_N}\left(Gm\mu\cos\xi_p - A_N\right)\sin\xi_p \ . \end{aligned} \tag{145}$$

Similarly, the evolutions $\dot{E}_N,\ \dot{L}_N$ and $\dot{A}_N$ [given by Eqs. (94)], can be rewritten in terms of ξ_p [by making use of the relations (12), (76) and (143)]:

$$\begin{aligned} r\dot{E}_N &= -\alpha Gm\mu\left(\frac{\mu}{-2E_N}\right)^{1/2}\sin\xi_p + \beta L_N\cos\xi_p \ , \\ \dot{L}_N &= \beta\frac{\mu\left(Gm\mu\cos\xi_p - A_N\right)}{-2E_N} - \alpha L_N\left(\frac{\mu}{-2E_N}\right)^{1/2}\sin\xi_p \ , \\ \dot{A}_N &= \beta L_N + \frac{L_N}{r}\left(\beta\frac{Gm\mu\cos\xi_p - A_N}{-2E_N} - \alpha\frac{L_N\sin\xi_p}{\left(-2\mu E_N\right)^{1/2}}\right)\cos\xi_p \end{aligned} \tag{146}$$

6.2 The evolution of the eccentric anomaly

In order to derive the perturbed Kepler equation for a given perturbing force with components $\alpha,\ \beta,\ \gamma$, we need the expression of $\dot{\xi}_p$ to first order accuracy. This can be found, following the logic of the derivation in the Keplerian case from subsection 2.4.4.

By taking the time derivative of Eq. (143)

$$-2E_N\dot{r} - 2\dot{E}_N r = -\dot{A}_N\cos\xi_p + A_N\dot{\xi}_p\sin\xi_p \ . \tag{147}$$

next employing the expressions (144) and (146) we obtain the desired relation

$$r\dot{\xi}_p = \left(\frac{-2E_N}{\mu}\right)^{1/2} + \beta\frac{Gm\mu L_N}{2E_N A_N}\sin\xi_p\cos\xi_p + \alpha\left(\frac{-2E_N}{\mu}\right)^{1/2}\frac{\mu}{A_N}\left[r^2 + \left(\frac{Gm\mu}{-2E_N}\right)^2\sin^2\xi_p\right] . \tag{148}$$

Another way of deriving this would be to start from the time derivative of the first Eq. (76), from which we eliminate $\sin\chi_p$ and $\cos\chi_p$ by use of the same Eqs. (76), to obtain

$$\dot{\xi}_p\cos\xi_p = \left(\frac{\dot{r}}{r} - \frac{\dot{L}_N}{L_N} + \frac{\dot{E}_N}{2E_N}\right)\sin\xi_p + \left(\frac{\mu}{-2E_N}\right)^{1/2}\frac{Gm\mu\cos\xi_p - A_N}{L_N}\dot{\chi}_p . \tag{149}$$

Next we employ once again the expressions (144)-(146) and we recover the relation (148).

6.3 The evolution of the Newtonian dynamical constants for ξ_p-dependent perturbing forces

By employing the eccentric anomaly parametrization for r, Eq. (143), the evolution of Eqs. (146) in time can be rewritten as evolution in ξ_p, with $dt = \dot{\xi}_p^{-1}d\xi_p$, and $\dot{\xi}_p$ given by Eq. (148). We also take $E_N = E_N^0 + E_N^1(\xi_p)$, $A_N = A_N^0 + A_N^1(\xi_p)$, $L_N = L_N^0 + L_N^1(\xi_p)$, the zeroth order terms being the constant Keplerian values. As all terms in Eqs. (146) are first order, we can employ the Newtonian expression of $\dot{\xi}_p$ [the leading order term in Eq. (148), $\dot{\xi}_p = (-2E_N/\mu)^{1/2}/r$]. We can also insert the constant values $E_N = E_N^0$, $A_N = A_N^0$, $L_N = L_N^0$, in the manifestly first order terms. We obtain:

$$\begin{aligned}
\left(\frac{-2E_N^0}{\mu}\right)^{1/2}\frac{dE_N^1}{d\xi_p} &= -\alpha Gm\mu\left(\frac{\mu}{-2E_N^0}\right)^{1/2}\sin\xi_p + \beta L_N^0\cos\xi_p , \\
\left(\frac{-2E_N^0}{\mu}\right)^{1/2}\frac{dL_N^1}{d\xi_p} &= r\left[\beta\frac{\mu\left(Gm\mu\cos\xi_p - A_N^0\right)}{-2E_N^0} - \alpha L_N^0\left(\frac{\mu}{-2E_N^0}\right)^{1/2}\sin\xi_p\right] , \\
\left(\frac{-2E_N^0}{\mu}\right)^{1/2}\frac{dA_N^1}{d\xi_p} &= \beta L_N^0 r \\
&\quad + L_N^0\left(\beta\frac{Gm\mu\cos\xi_p - A_N^0}{-2E_N^0} - \alpha\frac{L_N^0\sin\xi_p}{\left(-2\mu E_N^0\right)^{1/2}}\right)\cos\xi_p
\end{aligned} \tag{150}$$

Here r should be replaced by its expression (143), taken only to zeroth order accuracy, as it is everywhere multiplied by the components of the perturbing force. Provided the perturbing force components $\alpha,\ \beta$ can be expressed in terms of ξ_p alone, the above equations become ordinary differential equations, which formally integrate to

$$
\begin{aligned}
E_N &= E_N^0 - \frac{T_N^0}{\pi}\frac{E_N^0}{Gm\mu}\left(\frac{T_N^0}{\pi}E_N^0 I_{\alpha s} + L_N^0 I_{\beta c}\right) \ , \qquad (151)\\
L_N &= L_N^0 + \frac{T_N^0}{2\pi}\left\{\frac{\mu}{2E_N^0}\left[A_N^0\left(I_\beta + I_{\beta c^2}\right) - \frac{(Gm\mu)^2 + \left(A_N^0\right)^2}{Gm\mu}I_{\beta c}\right]\right.\\
&\qquad \left. + \frac{T_N^0}{\pi}\frac{E_N^0 L_N^0}{Gm\mu}\left(I_{\alpha s} - \frac{A_N^0}{Gm\mu}I_{\alpha sc}\right)\right\} \ , \qquad (152)\\
A_N &= A_N^0 + L_N^0\left[\frac{L_N^0}{2E_N^0}I_{\alpha sc} + \frac{T_N^0}{2\pi}\left(I_\beta + I_{\beta c^2} - 2\frac{A_N^0}{Gm\mu}I_{\beta c}\right)\right] \ . \qquad (153)
\end{aligned}
$$

with

$$
\begin{aligned}
I_{\alpha s} &= \int \alpha \sin\xi_p d\xi_p \ ,\\
I_{\alpha sc} &= \int \alpha \sin\xi_p \cos\xi_p d\xi_p \ ,\\
I_{\beta} &= \int \beta d\xi_p \ ,\\
I_{\beta c} &= \int \beta \cos\xi_p d\xi_p \ ,\\
I_{\beta c^2} &= \int \beta \cos^2\xi_p d\xi_p \ . \qquad (154)
\end{aligned}
$$

The notation was somewhat simplified by introducing the period T_N^0 of the unperturbed motion, defined by Eq. (46) in terms of E_N^0. As a consistency check, one can verify, that provided Eq. (12) holds for the Keplerian constants $E_N^0,\ L_N^0,\ A_N^0$ it will also hold for the values given in Eqs. (151)-(153).

From Eq. (151), by performing a series expansion in the small quantities $I_{\alpha s}$ and $I_{\beta c}$, we can also write the expression of the period T_N (of the Keplerian motion on the osculating ellipse), defined in terms of E_N in the same way as T_N^0 in terms of E_N^0:

$$
T_N = T_N^0\left[1 + \frac{3T_N^0}{2\pi Gm\mu}\left(\frac{T_N^0}{\pi}E_N^0 I_{\alpha s} + L_N^0 I_{\beta c}\right)\right] \ . \qquad (155)
$$

6.3.1 A note on the circularity of the perturbed orbit

In order the perturbed orbit to be circular (in the plane of motion, which can evolve due to γ), the eccentricity derived from $A_N = Gm\mu e$ should vanish. Therefore

$$
\frac{L_N^0}{-2E_N^0}I_{\alpha sc} + \frac{T_N^0}{2\pi}\left(2\frac{A_N^0}{Gm\mu}I_{\beta c} - I_\beta - I_{\beta c^2}\right) = \frac{A_N^0}{L_N^0} \qquad (156)
$$

should hold. If we further want the unperturbed orbit to be also circular, then $A_N^0 = 0$, such that

$$\frac{L_N^0}{-2E_N^0} I_{\alpha sc} = \frac{T_N^0}{2\pi} \left(I_\beta + I_{\beta c^2} \right) \ . \tag{157}$$

Obviously, this is a condition very few perturbing forces would obey. Also, the circularity of the perturbed orbit should be understood in a broad sense: while condition (156) holds the plane of motion could still undergo a precessional motion.

6.4 The Kepler equation in the presence of perturbations

By formally integrating Eq. (148), we obtain the perturbed Kepler equation.

We first rewrite Eq. (148), by employing a series expansion in the small parameters α and β as

$$\begin{aligned} \frac{dt}{d\xi_p} &= r \left(\frac{\mu}{-2E_N} \right)^{1/2} \left\{ 1 + \beta \left(\frac{\mu}{-2E_N} \right)^{3/2} \frac{GmL_N}{A_N} \sin \xi_p \cos \xi_p \right. \\ &\qquad \left. - \alpha \frac{\mu}{A_N} \left[r^2 + \left(\frac{Gm\mu}{-2E_N} \right)^2 \sin^2 \xi_p \right] \right\} \ . \end{aligned} \tag{158}$$

Next we insert the eccentric anomaly parametrization $r(\xi_p)$, Eq. (143) and obtain:

$$\begin{aligned} \frac{dt}{d\xi_p} &= \frac{T_N}{2\pi} \left(1 - \frac{A_N}{Gm\mu} \cos \xi_p \right) \left\{ 1 + \beta \frac{T_N}{2\pi} \frac{L_N}{A_N} \sin \xi_p \cos \xi_p \right. \\ &\qquad \left. + \alpha \left(\frac{T_N}{2\pi} \right)^2 \frac{2E_N}{A_N} \left[\left(1 - \frac{A_N}{Gm\mu} \cos \xi_p \right)^2 + \sin^2 \xi_p \right] \right\} \ . \end{aligned} \tag{159}$$

In order to simplify the notation, here we have employed again the definition of the Keplerian period, Eq. (46). In the perturbative terms the quantities $T_N,\ E_N,\ L_N\ ,\ A_N$ can be considered constants and replaced by their unperturbed values, however for the leading order term we need the evolutions of $A_N(\xi_p)$ and $T_N(\xi_p)$, given by Eqs. (153) and (155).

The Kepler equation is given then by the integral of Eq. (159):

$$\begin{aligned}
\frac{2\pi}{T_N^0}\left(t-t_0\right) &= \xi_p - \frac{A_N^0}{Gm\mu}\sin\xi_p - \frac{\left(L_N^0\right)^2}{2Gm\mu E_N^0}\int I_{\alpha sc}\cos\xi_p d\xi_p \\
&+\left(\frac{T_N^0}{2\pi}\right)^2\frac{6E_N^0}{Gm\mu}\left(\int I_{\alpha s}d\xi_p - \frac{A_N^0}{Gm\mu}\int I_{\alpha s}\cos\xi_p d\xi_p\right) \\
&+\frac{T_N^0}{2\pi}\frac{L_N^0}{Gm\mu}\left[3\int I_{\beta c}d\xi_p - \int\left(I_\beta + I_{\beta c^2} + \frac{A_N^0}{Gm\mu}I_{\beta c}\right)\cos\xi_p d\xi_p\right] \\
&+\frac{T_N^0}{2\pi}\frac{L_N^0}{A_N^0}\left(I_{\beta sc} - \frac{A_N^0}{Gm\mu}I_{\beta sc^2}\right) + \left(\frac{T_N^0}{2\pi}\right)^2\frac{2E_N^0}{A_N^0}\left\{2I_\alpha - \frac{4A_N^0}{Gm\mu}I_{\alpha c}\right. \\
&\left.-\left[1-3\left(\frac{A_N^0}{Gm\mu}\right)^2\right]I_{\alpha c^2} + \frac{A_N^0}{Gm\mu}\left[1-\left(\frac{A_N^0}{Gm\mu}\right)^2\right]I_{\alpha c^3}\right\}\ . \qquad (160)
\end{aligned}$$

with

$$\begin{aligned}
I_\alpha &= \int \alpha d\xi_p\ , \\
I_{\alpha c} &= \int \alpha\cos\xi_p d\xi_p\ , \\
I_{\alpha c^2} &= \int \alpha\cos^2\xi_p d\xi_p\ , \\
I_{\alpha c^3} &= \int \alpha\cos^3\xi_p d\xi_p\ , \\
I_{\beta sc} &= \int \beta\sin\xi_p\cos\xi_p d\xi_p\ , \\
I_{\beta sc^2} &= \int \beta\sin\xi_p\cos^2\xi_p d\xi_p \qquad (161)
\end{aligned}$$

As the integrations were formally carried out, the perturbed Kepler equation

(160) can be rewritten in terms of $T_N,\ E_N,\ L_N,\ A_N$:

$$
\begin{aligned}
\frac{2\pi}{T_N}\left(t-t_0\right) &= \xi_p - \frac{A_N}{Gm\mu}\sin\xi_p + \frac{L_N}{2Gm\mu}\left[\frac{L_N}{E_N}I_{\alpha sc}\right.\\
&\left.+3\left(\frac{T_N}{\pi}\right)^2\frac{E_N}{L_N}\frac{A_N}{Gm\mu}I_{\alpha s} + \frac{T_N}{\pi}\left(I_\beta + I_{\beta c^2} + \frac{A_N}{Gm\mu}I_{\beta c}\right)\right]\sin\xi_p\\
&-\frac{3T_N}{2\pi Gm\mu}\left(\frac{T_N}{\pi}E_N I_{\alpha s} + L_N I_{\beta c}\right)\xi_p - \frac{L_N^2}{2Gm\mu E_N}\int I_{\alpha sc}\cos\xi_p d\xi_p\\
&+\left(\frac{T_N}{2\pi}\right)^2\frac{6E_N}{Gm\mu}\left(\int I_{\alpha s}d\xi_p - \frac{A_N}{Gm\mu}\int I_{\alpha s}\cos\xi_p d\xi_p\right)\\
&+\frac{T_N}{2\pi}\frac{L_N}{Gm\mu}\left[3\int I_{\beta c}d\xi_p - \int\left(I_\beta + I_{\beta c^2} + \frac{A_N}{Gm\mu}I_{\beta c}\right)\cos\xi_p d\xi_p\right]\\
&+\frac{T_N}{2\pi}\frac{L_N}{A_N}\left(I_{\beta sc} - \frac{A_N}{Gm\mu}I_{\beta sc^2}\right) + \left(\frac{T_N}{2\pi}\right)^2\frac{2E_N}{A_N}\left\{2I_\alpha - \frac{4A_N}{Gm\mu}I_{\alpha c}\right.\\
&\left.-\left[1-3\left(\frac{A_N}{Gm\mu}\right)^2\right]I_{\alpha c^2} + \frac{A_N}{Gm\mu}\left[1-\left(\frac{A_N}{Gm\mu}\right)^2\right]I_{\alpha c^3}\right\}\ . \qquad (162)
\end{aligned}
$$

In order to evaluate the integrals appearing in either form of the Kepler equation, we need the explicit time-dependence of the perturbing force components, α and β. Examples for this will be discussed in the forthcoming sections.

We also note, that the constants of integration should be chosen in such a way, that the perturbative terms in the expressions of $E_N,\ L_N,\ A_N,\ T_N$ [given by Eqs. (151)-(153), (155)] and in either form of the Kepler equation [Eqs. (160), (162)] vanish at $\xi_p = 0$. In practice this means that all integrals (154), (161) should vanish at $\xi_p = 0$.

6.4.1 The radial period

In the perturbed two-body problem we define the radial period T of the perturbed motion as double the time elapsed between successive passages through points characterized by $\dot{r} = 0$. From Eqs. (144) we see that $\dot{r} = 0$ is equivalent to $\xi_p = 0, \pm\pi, \pm 2\pi$, etc., therefore the radial period is obtained as $T = \int_0^{2\pi} t\left(\xi_p\right) d\xi_p$, or more simply, by inserting $\xi_p = 2\pi$ in the perturbed Kepler equation derived in the previous subsection. We note that $T \neq T_N|_{\xi_p=2\pi}$ (the latter representing the radial period on the osculating ellipse at 2π).

6.5 Summary

In the present and the previous chapters we have developed the formalism suitable to describe the perturbed two-body problem. We have given explicit expressions in the parameter ξ_p for the dynamical constants of the osculating ellipse E_N and A_N [Eqs. (151) and (153)], which define the semimajor axis and the eccentricity of the osculating ellipse. The position of the osculating ellipse in the plane of motion is characterized by the precession of $\hat{\mathbf{A}}_{\mathbf{N}}$ about $\hat{\mathbf{L}}_{\mathbf{N}}$, or $\Omega_3\,(\chi_p)$ [see its expression in Eq. (100), which can be rewritten in terms of ξ_p if required]. Finally, the plane of motion is given by $\hat{\mathbf{L}}_{\mathbf{N}}$, precessing about $\hat{\mathbf{r}}$ [cf. Eq. (88)] in the case when the perturbing force has a component perpendicular to the plane of motion. As both precessions are first order effects, the directions about which the basis vectors precess, can be considered constants, defined by the unperturbed dynamics.

The dynamics of the perturbed two-body problem is entirely contained in the perturbed Kepler equation [in either of its forms (160) and (162)].

In the forthcoming chapters we will present some applications of the developed formalism.

7 Constant perturbing force

In this section we present an application of the true and eccentric anomaly parametrizations of the perturbed Keplerian motion. We consider bounded orbits only. In the computationally simplest toy model presented here the perturbing force is supposed to be constant on the orbit.

7.1 The evolution of the dynamical constants and of the radial period

The integrals (154), (161) become:

$$\begin{aligned}
I_{\alpha} &= \alpha\xi_p \ , \\
I_{\alpha c} &= \alpha\sin\xi_p \ , \\
I_{\alpha c^2} &= \frac{\alpha}{4}\left(2\xi_p+\sin 2\xi_p\right) \ , \\
I_{\alpha c^3} &= \frac{\alpha}{12}\left(\sin 3\xi_p+9\sin\xi_p\right) \ , \\
I_{\alpha s} &= \alpha\left(1-\cos\xi_p\right) \ , \\
I_{\alpha sc} &= \frac{\alpha}{4}\left(1-\cos 2\xi_p\right) \ , \\
I_{\alpha sc^2} &= \frac{\alpha}{12}\left(4-3\cos\xi_p-\cos 3\xi_p\right) \ .
\end{aligned} \tag{163}$$

The constants of integration were chosen such that at $\xi_p = 0$ all these integrals vanish. Similar relations hold for the corresponding integrals containing β.

The expressions (151)-(153), (155) give for the perturbed values $E_N,\ L_N,\ A_N\ ,\ T_N$:

$$E_N = E_N^0 - \frac{T_N^0}{\pi}\frac{E_N^0}{Gm\mu}\left(\alpha\frac{T_N^0}{\pi}E_N^0\left(1-\cos\xi_p\right)+\beta L_N^0\sin\xi_p\right) \ , \tag{164}$$

$$\begin{aligned}
L_N &= L_N^0 + \frac{T_N^0}{2\pi}\left\{\alpha\frac{T_N^0}{\pi}\frac{E_N^0 L_N^0}{4Gm\mu}\left(4-\frac{A_N^0}{Gm\mu}-4\cos\xi_p+\frac{A_N^0}{Gm\mu}\cos 2\xi_p\right)\right. \\
&\quad \left. +\beta\frac{\mu A_N^0}{8E_N^0}\left[6\xi_p-\frac{4Gm\mu}{A_N^0}\left[1+\left(\frac{A_N^0}{Gm\mu}\right)^2\right]\sin\xi_p+\sin 2\xi_p\right]\right\} \ ,
\end{aligned} \tag{165}$$

$$\begin{aligned}
A_N &= A_N^0 + \frac{L_N^0}{8}\left[\alpha\frac{L_N^0}{E_N^0}\left(1-\cos 2\xi_p\right)\right. \\
&\quad \left. +\beta\frac{T_N^0}{\pi}\left(6\xi_p-\frac{8A_N^0}{Gm\mu}\sin\xi_p+\sin 2\xi_p\right)\right] \ ,
\end{aligned} \tag{166}$$

$$T_N = T_N^0\left[1+\frac{3T_N^0}{2\pi Gm\mu}\left(\alpha\frac{T_N^0}{\pi}E_N^0\left(1-\cos\xi_p\right)+\beta L_N^0\sin\xi_p\right)\right] \ . \tag{167}$$

At $\xi_p = 0$ they take the initial values $E_N^0,\ L_N^0,\ A_N^0\ ,\ T_N^0$. Therefore, as a result of our choice of the integration constants, these unperturbed values characterize the osculating ellipse at the periastron.

By employing the relations (76) and (18) we find to leading order

$$\begin{aligned}\cos\xi_p &= \frac{A_N + Gm\mu\cos\chi_p}{Gm\mu + A_N\cos\chi_p}\ ,\\ \sin\xi_p &= \left(-\frac{2E_N}{\mu}\right)^{1/2}\frac{L_N\sin\chi_p}{Gm\mu + A_N\cos\chi_p}\ ,\end{aligned}\tag{168}$$

which allow to rewrite $E_N,\ L_N,\ A_N\ ,\ T_N$ in terms of the true anomaly χ_p. However the emerging expressions are more complicated than Eqs. (164)-(167).

7.1.1 Changes and averages over one radial period

We define the change of a quantity $f(\xi_p)$ over a radial period by $\Delta f(\xi_p) = f(\xi_p + 2\pi) - f(\xi_p)$. Such a change will be called a secular effect. Over a radial period we find

$$\Delta E_N(\xi_p) = \Delta T_N(\xi_p) = 0\ ,\tag{169}$$

$$\Delta L_N = \beta T_N^0 \frac{3\mu A_N^0}{4E_N^0}\ ,\tag{170}$$

$$\Delta A_N = \beta T_N^0 \frac{3L_N^0}{2}\ .\tag{171}$$

(The computation is very simple by realizing that we can drop all periodic and constant terms.)

We define the average of a quantity $f(\xi_p)$ over a radial period in the parameter ξ_p as $\overline{f}^{\xi} = (1/2\pi)\int_0^{2\pi} f(\xi_p)\,d\xi_p$. We find:

$$\begin{aligned}\overline{E_N}^{\xi} &= E_N^0\left[1-\alpha\left(\frac{T_N^0}{\pi}\right)^2\frac{E_N^0}{Gm\mu}\right]\ ,\\ \overline{L_N}^{\xi} &= L_N^0\left\{1+\frac{T_N^0}{2\pi}\left[\alpha\frac{T_N^0}{\pi}\frac{E_N^0}{4Gm\mu}\left(4-\frac{A_N}{Gm\mu}\right)+\beta\frac{3\pi\mu A_N^0}{4E_N^0L_N^0}\right]\right\}\ ,\\ \overline{A_N}^{\xi} &= A_N^0\left[1+\frac{L_N^0}{8A_N^0}\left(\alpha\frac{L_N^0}{E_N^0}+6\beta T_N^0\right)\right]\ ,\\ \overline{T_N}^{\xi} &= T_N^0\left[1+\alpha\frac{3E_N^0}{2Gm\mu}\left(\frac{T_N^0}{\pi}\right)^2\right]\ .\end{aligned}\tag{172}$$

Note that due to the secular contributions in β the definition above assures a correct average only for the first period. Therefore the values of L_N^0 and A_N^0 should be updated after each revolution. Alternatively, we can define the average over an arbitrary period starting at ξ_p^{init} by imposing the integration limits ξ_p^{init} and $\xi_p^{init} + 2\pi$. We also note that a similar definition in the parameter χ_p by $\overline{f} = (1/2\pi)\int_0^{2\pi} f(\chi_p)\,d\chi_p$ will give different averages, as the two parameters do not run with the same rate (although they take the same values in the periastron and apastron).

7.2 The Kepler equation

The additional integrals needed in the Kepler equation are:

$$\begin{aligned}
\int I_{\alpha s} d\xi_p &= \alpha\left(\xi_p - \sin\xi_p\right) \ , \\
\int I_{\beta c} d\xi_p &= \beta\left(1 - \cos\xi_p\right) \ , \\
\int I_{\alpha s}\cos\xi_p d\xi_p &= -\frac{\alpha}{4}\left(2\xi_p - 4\sin\xi_p + \sin 2\xi_p\right) \ , \\
\int I_{\beta c}\cos\xi_p d\xi_p &= \frac{\beta}{4}\left(1 - \cos 2\xi_p\right) \ , \\
\int \left(I_{\beta} + I_{\beta c^2}\right)\cos\xi_p d\xi_p &= \frac{\beta}{24}\left(-32 + 33\cos\xi_p - \cos 3\xi_p + 36\xi_p\sin\xi_p\right) \ , \\
\int I_{\alpha sc}\cos\xi_p d\xi_p &= \frac{\alpha}{24}\left(3\sin\xi_p - \sin 3\xi_p\right) \ . \qquad (173)
\end{aligned}$$

The Kepler equation (160) in the presence of a constant perturbing force becomes

$$\begin{aligned}
\frac{2\pi}{T_N^0}\left(t - t_0\right) &= \xi_p - \frac{A_N^0}{Gm\mu}\sin\xi_p + \alpha\left(\frac{T_N^0}{2\pi}\right)^2 \frac{E_N^0}{4Gm\mu} \\
&\times\left\{12\left(2 + \frac{Gm\mu}{A_N^0} + \frac{2A_N^0}{Gm\mu}\right)\xi_p - \left[51 + \frac{24A_N^0}{Gm\mu} + 5\left(\frac{A_N^0}{Gm\mu}\right)^2\right]\sin\xi_p \right. \\
&\left. -2\left(\frac{Gm\mu}{A_N^0} - \frac{6A_N^0}{Gm\mu}\right)\sin 2\xi_p + \left[1 - \left(\frac{A_N^0}{Gm\mu}\right)^2\right]\sin 3\xi_p\right\} \\
&+\beta\frac{T_N^0}{2\pi}\frac{L_N^0}{8Gm\mu}\left[2\left(16 + \frac{Gm\mu}{A_N^0} - \frac{A_N^0}{Gm\mu}\right) - 33\cos\xi_p \right. \\
&\left. +2\left(\frac{A_N^0}{Gm\mu} - \frac{Gm\mu}{A_N^0}\right)\cos 2\xi_p + \cos 3\xi_p - 12\xi_p\sin\xi_p\right] \ . \qquad (174)
\end{aligned}$$

For $\xi_p = 2\pi$ the elapsed time interval $t - t_0$ is the radial period T of the perturbed orbit:

$$T = T_N^0\left[1 + \alpha\left(\frac{T_N^0}{2\pi}\right)^2 \frac{3E_N^0}{Gm\mu}\left(2 + \frac{Gm\mu}{A_N^0} + \frac{2A_N^0}{Gm\mu}\right)\right] \ . \qquad (175)$$

Note that this is *not* the period of the osculating ellipse at $\xi_p = 2\pi$, found from Eq. (167) as $T_N|_{\xi_p=2\pi} = T_N^0$, neither is its angular average over the orbit given in Eq. (172). However for perturbing forces with $\alpha = 0$ all these coincide: $T = \overline{T_N}^{\xi} = T_N|_{\xi_p=2\pi} = T_N^0$.

The Kepler equation expressed in terms of the perturbed quantities, Eq.

(162) is somewhat simpler:

$$\begin{aligned}
\frac{2\pi}{T_N}\left(t-t_0\right) &= \xi_p - \frac{A_N}{Gm\mu}\sin\xi_p \\
&+\alpha\left(\frac{T_N}{2\pi}\right)^2\frac{E_N}{2Gm\mu}\left\{6\left(\frac{Gm\mu}{A_N}+\frac{2A_N}{Gm\mu}+2\cos\xi_p\right)\xi_p\right. \\
&\left.-4\left[6+\left(\frac{A_N}{Gm\mu}\right)^2\right]\sin\xi_p-\frac{Gm\mu}{A_N}\sin 2\xi_p\right\} \\
&+\beta\frac{T_N}{2\pi}\frac{L_N}{4Gm\mu}\left[\left(16+\frac{Gm\mu}{A_N}+\frac{A_N}{Gm\mu}\right)-16\cos\xi_p\right. \\
&\left.-\left(\frac{A_N}{Gm\mu}+\frac{Gm\mu}{A_N}\right)\cos 2\xi_p-12\xi_p\sin\xi_p\right] \ , \qquad (176)
\end{aligned}$$

however as here $T_N = T_N\left(\xi_p\right)$, it is less useful that Eq. (174). Still, it can be used to verify the expression of the radial period derived earlier. Indeed, for $\xi_p = 2\pi$, as $T_N|_{\xi_p=2\pi} = T_N^0$, it reproduces Eq. (175).

7.3 The periastron shift

The periastron shift is given by the precession of $\hat{\mathbf{A}}_{\mathbf{N}}$ about $\hat{\mathbf{L}}_{\mathbf{N}}$, thus by the Ω_3 component. This includes only contributions from the perturbing force components in the plane of motion, α and β. The periastron shift over one period T is defined as

$$\Delta\psi_0 = \int_0^T \Omega_3 dt \ . \qquad (177)$$

Here the period is defined as the time over which the system evolves from $r_{\min}$ through $r_{\max}$ into $r_{\min}$ again, thus double the time between two consecutive $\dot{r} = 0$ passages. As the integrand is already first order, the period can be safely chosen as $T = T_N$ (taking into account the difference $\Delta T = T - T_N$ would generate second order contributions).

Similarly, we can freely omit (or reinsert) the superscript ^{0} from (into) the dynamical quantities involved in the first order expressions.

7.3.1 Calculation based on the true anomaly

We can rewrite the integral (177) as an integral over χ_p:

$$\Delta\psi_0 = \int_0^{2\pi} \frac{\Omega_3\left(r\left(\chi_p\right),\chi_p\right)}{\dot{\chi}_p\left(r\left(\chi_p\right)\right)} d\chi_p \ . \qquad (178)$$

As Ω_3 is already of first order, the periastron precession angle to leading (first) order in the perturbations is found by employing the Newtonian expressions Eqs. (18) and (21), with χ_p in place of χ. If the components of the perturbing force are constants, as assumed, the integration can be carried out without the explicit knowledge of the perturbing force components:

$$\begin{aligned}\Delta\psi_0 &= -\int_0^{2\pi}\left[\alpha\frac{\mu r^2}{A_N}+(\alpha\sin\chi_p-\beta\cos\chi_p)\frac{Gm\mu^3r^3\sin\chi_p}{L_N^2A_N}\right]d\chi_p \\ &= -\frac{L_N^4}{G^2m^2\mu^3A_N}\left[\alpha\int_0^{2\pi}\frac{1+\sin^2\chi_p+\frac{A_N}{Gm\mu}\cos\chi_p}{\left(1+\frac{A_N}{Gm\mu}\cos\chi_p\right)^3}d\chi_p\right. \\ &\qquad\left.-\beta\int_0^{2\pi}\frac{\sin\chi_p\cos\chi_p}{\left(1+\frac{A_N}{Gm\mu}\cos\chi_p\right)^3}d\chi_p\right]\ . \end{aligned} \tag{179}$$

Although not impossible, these integrals are not particularly attractive to evaluate, due to the negative powers of $(Gm\mu + A_N\cos\chi_p)$, therefore we will try an alternative method for evaluating $\Delta\psi_0$.

7.3.2 Calculation based on the eccentric anomaly

An easier way to compute $\Delta\psi_0$ to first order accuracy over the period T would be to rewrite the integral (177) in terms of the eccentric anomaly:

$$\Delta\psi_0 = \int_0^{2\pi}\frac{\Omega_3\left(r\left(\xi_p\right),\xi_p\right)}{\dot{\xi}_p\left(r\left(\xi_p\right)\right)}d\xi_p\ . \tag{180}$$

The precessional angular velocity $\Omega_3\left(r\left(\xi_p\right),\xi_p\right)$ can be rewritten in terms of ξ_p by employing Eqs. (76) in the corresponding element (100). It becomes:

$$\begin{aligned}\Omega_3\left(r\left(\xi_p\right),\xi_p\right) &= -\alpha\frac{L_N}{A_N}-\left[\alpha L_N\sin\xi_p-\beta\left(\frac{\mu}{-2E_N}\right)^{1/2}(Gm\mu\cos\xi_p-A_N)\right] \\ &\quad\times\frac{Gm\mu\sin\xi_p}{-2E_NA_Nr}\ .\end{aligned} \tag{181}$$

We have not yet determined the rate of change of the eccentric anomaly $\dot{\xi}_p$ to first order accuracy, nevertheless $\dot{\xi}_p$ is needed in the expression (180) only to leading order, such that we can employ Eq. (56) with $\dot{\xi}_p$ in place of $\dot{\xi}$.

Therefore the integrand in Eq. (180) reads

$$\begin{aligned}\frac{d\psi_0}{d\xi} &= -\alpha\frac{GmL_N}{A_N}\left(\frac{\mu}{-2E_N}\right)^{3/2}\left(1-\frac{A_N}{Gm\mu}\cos\xi_p+\sin^2\xi_p\right) \\ &\quad+\beta\frac{G^2m^2\mu^3}{4A_NE_N^2}\left(\cos\xi_p-\frac{A_N}{Gm\mu}\right)\sin\xi_p\ ,\end{aligned} \tag{182}$$

which is straightforward to integrate, such that the periastron shift is found as

$$\Delta\psi_0 = -\alpha\frac{3\pi GmL_N}{A_N}\left(\frac{\mu}{-2E_N}\right)^{3/2}\ . \tag{183}$$

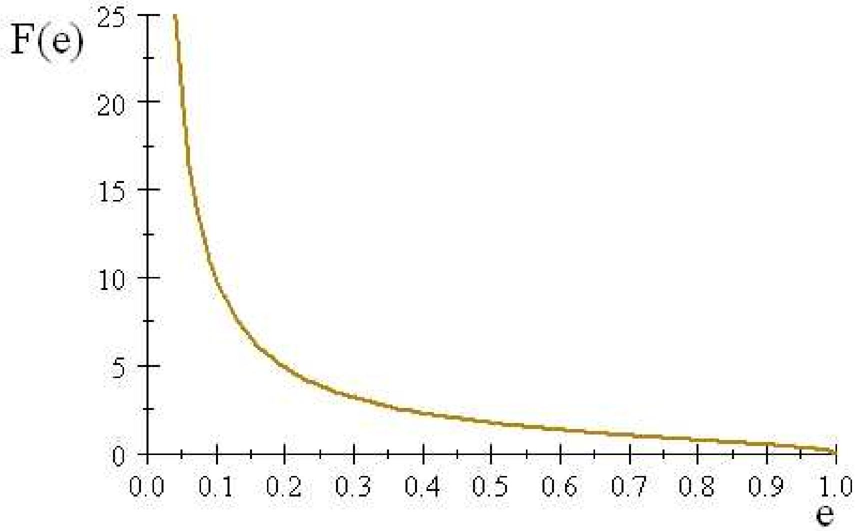

Figure 3: The form factor as function of eccentricity.

The periastron precession given in terms of orbital elements is

$$\Delta\psi_0 = -\alpha \frac{3\pi a^2 F(e)}{Gm} \ , \tag{184}$$

with the form factor

$$F(e) = \frac{\left(1 - e^2\right)^{1/2}}{e} \tag{185}$$

represented on Fig. 3. Note that the β-dependence dropped out from the expression of $\Delta\psi_0$ taken over one period, therefore *from among the constant perturbing forces those giving secular periastron precession are the ones oriented towards the periastron.*

7.3.3 Average precession rate

As an example of how we compute a secular effect (the average over one orbit of the instantaneous effect), we compute the average of the precessional angular velocity of the periastron, $\Omega_3 \equiv d\psi_0/dt$. This can be found from the general prescription of calculating an average over one orbit:

$$\langle \Omega_3 \rangle \equiv \frac{1}{T_N} \int_0^{T_N} \Omega_3 dt = \frac{\Delta\psi_0}{T_N} \ . \tag{186}$$

By employing Eq. (46) for the radial period, we find

$$\langle \Omega_3 \rangle = -\alpha \frac{3L_N}{2A_N} = -\alpha \frac{3}{2e} \left[\frac{a\left(1-e^2\right)}{Gm} \right]^{1/2} \ .$$

7.4 Comment on the parametrizations

In all computations considered in this chapter so far the eccentric anomaly parametrization turned out to be more suitable in obtaining the results, than the true anomaly parametrization. The periastron precession, the expressions of the Newtonian dynamical constants on an arbitrary point of the orbit and the Kepler equation were all obtained employing the eccentric anomaly parametrization.

Let us discuss, why. In the derivation of the periastron precession the integrand Ω_3 is a sum of a $\sin\,\xi_p$, a $\sin\,^2\xi_p$ and a $\sin\,\xi_p \cos\xi_p$ terms, Eq. (181). As Ω_3 is already first order, when passing to the integration variable ξ_p the integrand should be multiplied by the zeroth order expression of $\dot{\xi}_p^{-1} \propto r$, which again is a polynomial in $\cos\xi_p$. This makes the integrand in the variable ξ_p a simple polynomial in $\sin\,\xi_p$ and $\cos\xi_p$, straightforward to integrate.

By comparison, expressed in terms of the true anomaly parametrization, Ω_3 is a sum of a constant, an $r\sin^2\chi_p$ and an $r\sin\chi_p\cos\chi_p$ terms, see Eq. (100). By passing to χ_p as an integration variable, it should be multiplied by the zeroth order expression of $\dot{\chi}^{-1} \propto r^2 \propto (Gm\mu + A_B\cos\chi_p)^{-2}$. The integrand is not a simple polynomial any more, and it becomes cumbersome to integrate.

In the computation leading to the explicit expressions of the dynamical constants of the osculating ellipse a similar situation was encountered: the integrands were simple polynomials in trigonometric functions of ξ_p, while rewritten in χ_p they would contain negative powers of $(Gm\mu + A_B\cos\chi_p)$.

When deriving the Kepler equation (both in the perturbed and in the unperturbed cases), we have encountered a similar situation. In the unperturbed case the integrand was simply $\dot{\xi}_p^{-1} \propto r$, a polynomial in $\cos\xi_p$. A similar situation occurred in the perturbed case, as the perturbing force was chosen a constant.

However we can imagine situations, where the true anomaly parametrization is more useful. For any first order integrand $\propto r^{n\leq -2}$, when passing to the true anomaly parametrization, taken in the zeroth order (where it has the property $r^2\dot{\chi}_p = L_N/\mu$), the new integrand will become a polynomial in $\sin\chi_p$ and $\cos\chi_p$, easy to integrate. Conversely, for any first order integrand $\propto r^{n\geq -1}$, when passing to the eccentric anomaly parametrization, taken in the zeroth order (where it has the property $r\dot{\xi}_p = (-2E_N/\mu)^{1/2}$), the new integrand will become a polynomial in $\sin \xi_p$ and $\cos \xi_p$, again easy to integrate.

These convenient properties hold for any Keplerian calculation and for perturbed Keplerian calculations, when evaluating the evolution of first order quantities over the orbit, as in the example of periastron shift for constant perturbing forces. However, when evaluating zeroth order quantities over the orbit, the advantageous property of the parametrizations is lost, as shown by Eqs. (134) and (100) for $\dot{\chi}_p$ and Eq. (148) for $\dot{\xi}_p$. In some cases, like the computations of this chapter, based on the assumption of a constant perturbing force, the computations can still be done relatively easy. In other cases the integrals may become cumbersome (although not impossible to evaluate). This is the price one has to pay for extending the definition of the true and eccentric anomaly parametrizations of r derived for the Keplerian motion in an unmodified form for the perturbed Keplerian dynamics.

As an alternative, we could introduce *modified* true and eccentric anomaly parametrizations, reducing to the Keplerian definitions in the absence of perturbations, such that we their expressions $\dot{\chi}$ and $\dot{\xi}$ coincide with the Keplerian ones, and the nice integrability features are conserved.

7.5 Secular orbital evolution in the plane of motion

Due to Eq. (169) the semimajor axis of the binary remains unchainged under the action of the constant perturbative force. The eccentricity and the periastron change according to Eqs. (171) and (183) as

$$\frac{\Delta e}{e} = \frac{\Delta A_N}{A_N^0} = \frac{\beta}{\varkappa_{\hat{\mathbf{L}}_\mathbf{N}}} \ , \qquad (187)$$

$$\frac{\Delta \psi_0}{2\pi} = -\frac{\alpha}{\varkappa_{\hat{\mathbf{L}}_\mathbf{N}}} \ , \qquad (188)$$

where $\varkappa_{\hat{\mathbf{L}}_\mathbf{N}}$ is a constant with dimension of acceleration (the subscript $_{\hat{\mathbf{L}}_\mathbf{N}}$ is for rotation about $\hat{\mathbf{L}}_\mathbf{N}$):

$$\varkappa_{\hat{\mathbf{L}}_\mathbf{N}} \equiv \frac{2A_N^0}{3L_N^0 T_N^0} = \frac{2Gm}{3a_0^2 F\left(e_0\right)} \ . \qquad (189)$$

For a perturbing force perpendicular to the plane of motion therefore there is no secular evolution of the quasi-elliptic orbit in the plane of motion. In all other cases the relative change of the eccentricity and azimuthal angle at periastron scale with the respective components of the perturbative force.

The eccentricity changes only due to β, the conponent of the perturbative force in the plane of motion and perpendicular to the semimajor axis. For $\beta < 0$ the eccentricity decreases and the orbit tends to circularize. On the contrary, for $\beta > 0$ the eccentricity of the orbit is increased and the quasi-elliptic trajectory tends to become more elongated.

quadrant	direction of $\Delta\mathbf{a}_\perp$	$\frac{\Delta\psi_0}{2\pi} = -\frac{\alpha}{\varkappa}$	effect on the ellipse
1.	$\xi_p = \pi/4$	$-\varepsilon$	counter-rotation
	$(\alpha = \beta = \varepsilon\varkappa)$		$\hat{\mathbf{Q}}_\mathbf{N}$ tends to align to $\Delta\mathbf{a}_\perp$
2.	$\xi_p = 3\pi/4$	ε	co-rotation
	$(-\alpha = \beta = \varepsilon\varkappa)$		$\hat{\mathbf{Q}}_\mathbf{N}$ tends to align to $\Delta\mathbf{a}_\perp$
3.	$\xi_p = -3\pi/4$	ε	co-rotation
	$(\alpha = \beta = -\varepsilon\varkappa)$		$\hat{\mathbf{A}}_\mathbf{N}$ turns opposite to $\Delta\mathbf{a}_\perp$
4.	$\xi_p = -\pi/4$	$-\varepsilon$	counter-rotation
	$(\alpha = -\beta = \varepsilon\varkappa)$		$\hat{\mathbf{A}}_\mathbf{N}$ tends to align to $\Delta\mathbf{a}_\perp$

Table 2: The effect on the periastron of the specific perturbing force $\Delta\mathbf{a}$, for 4 particular configurations. The periastron co-rotates (counter-rotates) with μ for projections of the perturbing force in the second and third (fourth and first) quadrants, such that the semiminor axis tends to align to the projection to the plane of motion of the perturbing force $\Delta\mathbf{a}_\perp$.

The periastron co-rotates with the orbit ($\Delta\psi_0 > 0$), when $\alpha < 0$ and counter-rotates, when $\alpha > 0$.

Depending on which quadrant is the projection of the perturbing force, it will have different effects. Let us illustrate this with four particular configurations for the perturbing force, given by $|\alpha| = |\beta| = \varepsilon\varkappa$, where $\varepsilon > 0$ represents a small positive number, which scales the perturbation. In other worlds, the projection of the perturbing force onto the plane of motion for these configurations is at $\xi_p = \pm\pi/4, \ \pm 3\pi/4$. The effect of the perturbing force for these configurations is summarized in Tables 2, 3.

The properties found however for these particular configurations are characteristic for any force with components in the same quadrant (only the magnitude of the effects vary). The eccentricity increases for perturbing forces with projection in the plane of motion lying in the first and second quadrant; and decreases for perturbing forces with projections in the third and fourth quadrant. The periastron co-rotates (counter-rotates) with the motion of the mass μ for projections of the perturbing force in the second and third (fourth and first) quadrants.

If the projection of the perturbing force falls into the first or fourth quadrant, as the periastron counter-rotates in this case, the quasi-ellipse tends to turn such that $\hat{\mathbf{Q}}_\mathbf{N}$ becomes aligned to the original direction of the projection of the perturbing force $\Delta\mathbf{a}_\perp$ to the plane of motion. Thus in the basis ($\hat{\mathbf{A}}_\mathbf{N}$, $\hat{\mathbf{Q}}_\mathbf{N}$) the projection $\Delta\mathbf{a}_\perp$ of the specific perturbing force will turn towards the positive y-direction.

If the projection of the perturbing force falls into the second or third quadrant, as the periastron co-rotates in this case, the quasi-ellipse again tends to turn such that $\hat{\mathbf{Q}}_\mathbf{N}$ becomes aligned to $\Delta\mathbf{a}_\perp$. Therefore this configuration is stable.

In this stable configuration the eccentricity increases in time due to the

quadrant	direction of $\Delta\mathbf{a}_\perp$	$\frac{\Delta e}{e}=\frac{\beta}{\varkappa}$	effect on the ellipse
1.	$\xi_p=\pi/4$ $(\alpha=\beta=\varepsilon\varkappa)$	ε	elongation
2.	$\xi_p=3\pi/4$ $(-\alpha=\beta=\varepsilon\varkappa)$	ε	elongation
3.	$\xi_p=-3\pi/4$ $(\alpha=\beta=-\varepsilon\varkappa)$	$-\varepsilon$	circularization
4.	$\xi_p=-\pi/4$ $(\alpha=-\beta=\varepsilon\varkappa)$	$-\varepsilon$	circularization

Table 3: The effects on the eccentricity of the specific perturbing force projection $\Delta\mathbf{a}_\perp$, for 4 particular configurations. The orbit becomes more elongated for perturbing forces with projection in the plane of motion lying in the first and second quadrant; and it tends to circularize for perturbing forces with projections in the third and fourth quadrant. As in the preferred configuration the semiminor axis (the base vector $\hat{\mathbf{Q}}_\mathbf{N}$) of the orbit is aligned to the projection of the perturbing force (see main text), the ellipse becomes more elongated with each revolution, (keeping though its semimajor axis). This energy-conserving, but orbital angular momentum extracting process eventually results in the merger of the binary.

perturbing force. However, as stated earlier, the semimajor axis stays constant. Therefore, if there is sufficient time for the perturbing force to act, eventually the orbit will become so much flattenned ($r_{\rm min}=2a-r_{\rm max}\ll r_{\rm max}$), that the binary components will actually merge at the point of closest approach, rather than overpassing each other.

Therefore, *the constant perturbing force does not extract energy from the system,* however its projection into the plane of motion *extracts angular momentum.*

7.6 Secular evolution of the plane of motion

The evolution of the plane of motion is contained in Eq. (89). From here the shifts of $\hat{\mathbf{L}}_\mathbf{N}$ about $\hat{\mathbf{A}}_\mathbf{N}$ (along $-\hat{\mathbf{Q}}_\mathbf{N}$) and about $\hat{\mathbf{Q}}_\mathbf{N}$ (along $\hat{\mathbf{A}}_\mathbf{N}$) during one radial period are:

$$
\begin{aligned}
\Delta\phi_{\hat{\mathbf{A}}_\mathbf{N}} &= \frac{\gamma\mu}{L_N^0}\int_0^T r\cos\chi_p dt\ , \\
\Delta\phi_{\hat{\mathbf{Q}}_\mathbf{N}} &= \frac{\gamma\mu}{L_N^0}\int_0^T r\sin\chi_p dt\ . \qquad (190)
\end{aligned}
$$

Due to the simplicity of the integrands these calculations are much easier to perform than the respective calculation for the periastron shift. We pass to the integration in terms of the eccentric anomaly by $dt=\dot{\xi}_p^{-1}d\xi_p$, then we employ Eqs. (148), Eqs. (76), finally Eq. (143) (all needed only to leading order) and obtain:

$$\begin{aligned}
\Delta\phi_{\hat{\mathbf{A}}_{\mathbf{N}}} &= \gamma T_N^0 \frac{3\mu A_N^0}{4E_N^0 L_N^0} \ , \\
\Delta\phi_{\hat{\mathbf{Q}}_{\mathbf{N}}} &= 0 \ .
\end{aligned} \tag{191}$$

Therefore the plane of motion is rotated about $\hat{\mathbf{A}}_{\mathbf{N}}$ after each radial period by the angle $\Delta\phi_{\hat{\mathbf{A}}_{\mathbf{N}}}$. Remembering, that the positive sense of rotation about the basis vectors $\left\{\mathbf{f}_{(\mathbf{i})}\right\}$ is counter-clockwise, we find the following. The rotation is counter-clockwise ($\Delta\phi_{\hat{\mathbf{A}}_{\mathbf{N}}} > 0$) if the force and the orbital angular momentum span an obtuse angle ($\gamma < 0$), and clockwise, if they span an accute angle (such that $\gamma > 0$). Clearly, the stable configuration is reached when the perturbing force spans the right angle with the orbital angular momentum, such that the force becomes tangent to the plane of motion.

Thus, there is no evolution of the plane of motion for $\gamma = 0$, no evolution in the orientation of the plane of motion if $\alpha = 0$, and no change in the excentricity if $\beta = 0$.

In order to compare this rotational shift with the precession of the periastron, we rewrite $\Delta\phi_{\hat{\mathbf{A}}_{\mathbf{N}}}$ as

$$\frac{\Delta\phi_{\hat{\mathbf{A}}_{\mathbf{N}}}}{2\pi} = -\frac{\gamma}{\varkappa_{\hat{\mathbf{A}}_{\mathbf{N}}}} \ , \tag{192}$$

with

$$\varkappa_{\hat{\mathbf{A}}_{\mathbf{N}}} = \frac{2GmF\left(e_0\right)}{3a_0^2} = \varkappa_{\hat{\mathbf{L}}_{\mathbf{N}}} F^2\left(e_0\right) \ . \tag{193}$$

As $F\left(e_0\right)$ diverges in the origin and goes to zero at $e_0 = 1$, for small eccentricities $\varkappa_{\hat{\mathbf{L}}_{\mathbf{N}}} \ll \varkappa_{\hat{\mathbf{A}}_{\mathbf{N}}}$, while for high eccentricities $\varkappa_{\hat{\mathbf{L}}_{\mathbf{N}}} \gg \varkappa_{\hat{\mathbf{A}}_{\mathbf{N}}}$. They become equal at $e_0 = 2^{-1/2}$. The smaller the value of the coefficients $\varkappa_{\hat{\mathbf{L}}_{\mathbf{N}}}$, $\varkappa_{\hat{\mathbf{A}}_{\mathbf{N}}}$, the bigger the corresponding secular effect. For nearly circular orbits therefore the above rotation of the plane of motion about the periastron axis is much faster, than the periastron precession and the increase of eccentricity. This situaton is reversed for high eccentricities, where the periastron precession is fast and the ecentricity increases quickly.

We conclude that the effect of the constant perturbative force on a nearly circular orbit during one period consist of:

(a) Turning the plane of motion parallel to the force by a rotation about $\hat{\mathbf{A}}_{\mathbf{N}}$

(b) Turning the osculating ellipse such that the force points along $\hat{\mathbf{Q}}_{\mathbf{N}}$

(c) Increasing or decreasing the eccentricity, according to the sign of β.

While the process (a) is relatively fast for small eccentricities, (b) and (c) dominate at high values of the eccentricity. These determine the initial data for the next period, where the perturbing force is reoriented due to the assumption of α, β, γ being constants.

8 Equal mass binary perturbed by a small, center of mass located body

Let us assume that the two body problem is perturbed by a third mass $\delta m \ll \min(m_1, m_2)$, located in the center of mass of the two more massive bodies. Then the equations of motion are

$$\begin{aligned}
\ddot{\mathbf{r}}_1 &= \frac{Gm_2}{r^2}\mathbf{r} - \frac{G\delta m}{r_1^2}\mathbf{r}_1 \ , \\
\ddot{\mathbf{r}}_2 &= -\frac{Gm_1}{r^2}\mathbf{r} - \frac{G\delta m}{r_2^2}\mathbf{r}_2 \ , \\
\delta\ddot{\mathbf{r}} &= \frac{Gm_1}{r_1^2}\mathbf{r}_1 + \frac{Gm_2}{r_2^2}\mathbf{r}_2 \ . \qquad (194)
\end{aligned}$$

Employing Eqs. (5) and deriving the evolution for $\mathbf{r}$ one obtains

$$\ddot{\mathbf{r}} = -\frac{Gm}{r^2}\left(1 + \frac{\delta m}{\mu}\right)\mathbf{r} \ . \qquad (195)$$

The dynamics of the two main bodies is perturbed then by

$$\Delta\mathbf{a} = -\frac{Gm\delta m}{\mu r^2}\left(\frac{L_N^2 - Gm\mu^2 r}{\mu A_N}\hat{\mathbf{A}}_{\mathbf{N}} + \frac{L_N}{A_N} r\dot{r}\hat{\mathbf{Q}}_{\mathbf{N}}\right) , \qquad (196)$$

such that

$$\begin{aligned}
\alpha &= -\frac{Gm\delta m\left(L_N^2 - Gm\mu^2 r\right)}{\mu^2 A_N r^2} \ , \\
\beta &= -\frac{Gm\delta m L_N \dot{r}}{\mu A_N r} \ , \qquad \gamma = 0 \qquad (197)
\end{aligned}$$

Here we have used the expression of $\mathbf{r}$ given in Eqs. (32) and (34).

The evolution for $\delta\mathbf{r}$ is given by

$$\delta\ddot{\mathbf{r}} = \frac{Gm}{r^2}\left(\frac{m_2}{m_1} - \frac{m_1}{m_2}\right)\mathbf{r} \ .$$

In order to have δm in equilibrium, in what follows we take $m_2 = m_1$. Hence δm can be in equilibrium in the cemter of mass of the other two bodies, provided its initial velocity is also zero. The equilibrium is unstable, nevertheless it can be maintained, if we think of a spaceship with its own engine, positioned in the center of mass of a binary formed by two equal-mass neutron stars.

Exam work: describe in detail the perturbed Keplerian dynamics of the two larger bodies in the case of this perturbing force.

A Computation details for subsection 5.1

In this Appendix we use summation convention over repeated (covariant and contravariant) indices.

The velocity in the perturbed basis is:

$$\begin{aligned}
\mathbf{v} &= \dot{x}^i \mathbf{f_{(i)}} + x^i \dot{\mathbf{f}}_{(\mathbf{i})} = \dot{x}^i \mathbf{f_{(i)}} + x^i \mathbf{\Omega} \times \mathbf{f_{(i)}} \\
&= \dot{x}^i \mathbf{f_{(i)}} + x^i \epsilon^{sqp} \Omega_q \delta_{ip} \mathbf{f_{(s)}} \\
&= \dot{x}^i \mathbf{f_{(i)}} + x^p \epsilon^{sq}{}_{p} \Omega_q \mathbf{f_{(s)}} \\
&= \left(\dot{x}^i - \epsilon^i{}_p{}^q x^p \Omega_q\right) \mathbf{f_{(i)}} \ ,
\end{aligned} \tag{198}$$

where the coordinates are given by Eq. (129) and in consequence

$$\dot{x}^1 = \dot{r}\cos\chi_p - r\dot{\chi}_p \sin\chi_p \ , \qquad \dot{x}^2 = \dot{r}\sin\chi_p + r\dot{\chi}_p\cos\chi_p \ , \qquad \dot{x}^3 = 0 \ . \tag{199}$$

The orbital angular momentum becomes

$$\begin{aligned}
\frac{\mathbf{L_N}}{\mu} &= \epsilon^i{}_{pq} x^p \left(\dot{x}^q - \epsilon^q{}_j{}^k x^j \Omega_k\right) \mathbf{f_{(i)}} \\
&= \left[\epsilon^i{}_{pq} x^p \dot{x}^q - \left(\delta^i_j \delta^k_p - \delta_{jp}\delta^{ik}\right) x^p x^j \Omega_k\right] \mathbf{f_{(i)}} \\
&= \left[\left(x^1\dot{x}^2 - x^2\dot{x}^1\right)\delta^i_3 - \left(x^k x^i \Omega_k - x_j x^j \Omega_i\right)\right] \mathbf{f_{(i)}} \\
&= \left(r^2 \dot{\chi}_p \delta^i_3 - x^k x^i \Omega_k + x_j x^j \Omega_i\right) \mathbf{f_{(i)}} \\
&= \left(-x^k x^1 \Omega_k + x_j x^j \Omega_1\right) \mathbf{f_{(1)}} \\
&\quad + \left(-x^k x^2 \Omega_k + x_j x^j \Omega_2\right) \mathbf{f_{(2)}} \\
&\quad + \left(r^2 \dot{\chi}_p + x_j x^j \Omega_3\right) \mathbf{f_{(3)}} \\
&= \left(x^2 \Omega_1 - x^1 \Omega_2\right)\left(x^2 \mathbf{f_{(1)}} - x^1 \mathbf{f_{(2)}}\right) \\
&\quad + \left[r^2 \dot{\chi}_p + \left(x^1 x^1 + x^2 x^2\right)\Omega_3\right] \mathbf{f_{(3)}} \\
&= r^2 \left(\dot{\chi}_p + \Omega_3\right) \hat{\mathbf{L}}_{\mathbf{N}} \ ,
\end{aligned} \tag{200}$$

where we have employed $x^2\Omega_1 = x^1\Omega_2$, see Eq. (100).

For the calculation of the energy, we employ:

$$\begin{aligned}
v^2 &= \left(\dot{x}^i - \epsilon^i{}_j{}^k x^j \Omega_k\right)\left(\dot{x}^s - \epsilon^s{}_p{}^q x^p \Omega_q\right)\delta_{is} \\
&= \dot{x}^i \dot{x}^s \delta_{is} - 2\epsilon^i{}_j{}^k x^j \dot{x}^s \Omega_k \delta_{is} + \epsilon^i{}_p{}^q \epsilon_{ij}{}^k x^j x^p \Omega_k \Omega_q \\
&= \left(\dot{x}^1\right)^2 + \left(\dot{x}^2\right)^2 - 2\epsilon^1{}_j{}^k x^j \dot{x}^1 \Omega_k - 2\epsilon^2{}_j{}^k x^j \dot{x}^2 \Omega_k \\
&\quad + \left(x^1\right)^2 \Omega_2^2 + \left(x^2\right)^2 \Omega_1^2 + r^2 \Omega_3^2 - 2x^1 x^2 \Omega_2 \Omega_1 \\
&= \dot{r}^2 + r^2 \dot{\chi}_p^2 + 2r^2 \dot{\chi}_p \Omega_3 + r^2 \Omega_3^2 \ .
\end{aligned} \tag{201}$$

Remarkably, many of the second order terms in Ω_i-s cancelled out.

The Laplace-Runge-Lenz vector is found by a similar calculation:

$$
\begin{aligned}
\mathbf{A_N} &= \epsilon_i^{\ 3k}\left(\dot{x}^i - \epsilon^i_{\ m}{}^n x^m \Omega_n\right)\mu r^2\left(\dot{\chi}_p + \Omega_3\right)\mathbf{f_{(k)}} - \frac{Gm\mu}{r}x^k \mathbf{f_{(k)}} \\
&= \left[\mu r^2\left(\dot{\chi}_p + \Omega_3\right)\left(\dot{x}^2 + x^1\Omega_3\right) - \frac{Gm\mu}{r}x^1\right]\mathbf{\hat{A}_N} \\
&\quad + \left[-\mu r^2\left(\dot{\chi}_p + \Omega_3\right)\left(\dot{x}^1 - x^2\Omega_3\right) - \frac{Gm\mu}{r}x^2\right]\mathbf{\hat{Q}_N} \\
&= \left[\mu r^2\dot{\chi}_p\left(\dot{r}\sin\chi_p + r\dot{\chi}_p\cos\chi_p\right) - Gm\mu\cos\chi_p\right. \\
&\quad \left. + \mu r^2\Omega_3\left(\dot{r}\sin\chi_p + 2r\dot{\chi}_p\cos\chi_p + r\Omega_3\cos\chi_p\right)\right]\mathbf{\hat{A}_N} \\
&\quad - \left[\mu r^2\dot{\chi}_p\left(\dot{r}\cos\chi_p - r\dot{\chi}_p\sin\chi_p\right) + Gm\mu\sin\chi_p\right. \\
&\quad \left. + \mu r^2\Omega_3\left(\dot{r}\cos\chi_p - 2r\dot{\chi}_p\sin\chi_p - \Omega_3 r\sin\chi_p\right)\right]\mathbf{\hat{Q}_N}\ . \qquad (202)
\end{aligned}
$$

References

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