Ruijsenaars duality in the framework of symplectic reduction

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• Two integrable many-body systems are dual to each other if the action variables of system (i) are the particle positions of system (ii), and vice versa. Underlying phase spaces are symplectomorphic.

• First example is the self-duality of the rational Calogero system. Interpreted in terms of symplectic reduction by Kazhdan, Kostant and Sternberg (1978).

• Duality was discovered and explored by Ruijsenaars (1988-95) in his direct construction of action-angle variables for Calogero-Sutherland type systems and their 'relativistic' deformations.

The simplest example

Rational Calogero system: $H_{\text{Cal}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{(q_k - q_j)^2}$

Symplectic reduction: Consider phase space $T^*\mathrm{i}u(n) \simeq \mathrm{i}u(n) \times \mathrm{i}u(n) := \{(Q,P)\}$ with two families of 'free' Hamiltonians $\{\mathrm{tr}\,(Q^k)\}$ and $\{\mathrm{tr}\,(P^k)\}$. Reduce by the adjoint action of $U(n)$ using the moment map constraint

$$[Q,P] = \mu(x) := \mathrm{i}x \sum_{j\neq k} E_{j,k}$$

This yields the self-dual Calogero system (OP [76], KKS [78]):

gauge slice (i): $Q = q := \mathrm{diag}(q_1,\dots,q_n)$, $q_1 > \cdots > q_n$, with $p := \mathrm{diag}(p_1,\dots,p_n)$

$$P = p + \mathrm{i}x \sum_{j\neq k} \frac{E_{jk}}{q_j - q_k} \equiv L_{\text{Cal}}(q,p) \quad \text{Lax matrix}, \quad \mathrm{tr}\,(dP\wedge dQ) = \sum_{k=1}^{n} dp_k \wedge dq_k$$

gauge slice (ii): $P = \hat{p} := \mathrm{diag}(\hat{p}_1,\dots,\hat{p}_n)$, $\hat{p}_1 > \cdots > \hat{p}_n$, with $\hat{q} := \mathrm{diag}(\hat{q}_1,\dots,\hat{q}_n)$

$$Q = -L_{\text{Cal}}(\hat{p},\hat{q}) \quad \text{dual Lax matrix}, \quad \mathrm{tr}\,(dP\wedge dQ) = \sum_{k=1}^{n} d\hat{q}_k \wedge d\hat{p}_k.$$

The alternative gauge slices give two models of the reduced phase space. Their natural symplectomorphism is the 'action-angle map' for the two Calogero systems: alias the duality map. Ruijsenaars hinted at analogous picture in general.

A 'dual pair' of integrable many-body systems

Hyperbolic Sutherland system (1971):

$$H_{\text{hyp-Suth}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{x^2}{2}\sum_{j\neq k}\frac{1}{\sinh^2(q_j - q_k)}$$

Basic Poisson brackets: $\{q_i, p_j\} = \delta_{i,j}$, x: non-zero, real constant.

Rational Ruijsenaars-Schneider system (1986):

$$H_{\text{rat-RS}}(\hat{p},\hat{q}) = \sum_{k=1}^{n}\cosh(\hat{q}_k)\prod_{j\neq k}\left[1 + \frac{x^2}{(\hat{p}_k - \hat{p}_j)^2}\right]^{\frac{1}{2}}$$

Poisson brackets: $\{\hat{p}_i, \hat{q}_j\} = \delta_{i,j}$ ($\hat{p}_i$ are RS 'particle positions').

Systems describe n 'particles' moving on the line, and are integrable.

Ruijsenaars (1988) constructed 'duality symplectomorphism' (action-angle map) between underlying phase spaces.

Local description of two other dual pairs

Standard trigonometric Ruijsenaars-Schneider [86] system:

$$H_{\text{trigo}-\text{RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j\neq k} \left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

It is a relativistic generalization (here with $c = 1$) of

$$H_{\text{trigo}-\text{Suth}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{x^2}{2}\sum_{j\neq k} \frac{1}{\sin^2(q_k - q_j)}$$

The dual systems (Ruijsenaars [88,95]):

$$\widehat{H}_{\text{trigo}-\text{RS}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j\neq k} \left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_j)}\right]^{\frac{1}{2}}$$

$$\widetilde{H}_{\text{rat}-\text{RS}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j\neq k} \left[1 - \frac{x^2}{(\hat{p}_k - \hat{p}_j)^2}\right]^{\frac{1}{2}}$$

$H_{\text{trigo}-\text{RS}}$, $\widehat{H}_{\text{trigo}-\text{RS}}$: different real forms of complex trigo RS.

Further self-dual systems

Compactified trigonometric RS ($\mathrm{III_b}$) system, locally given by

$$H_{\text{compact-RS}} = \sum_{k=1}^{n} (\cos p_k) \prod_{j \neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)} \right]^{\frac{1}{2}}$$

Hyperbolic Ruijsenaars-Schneider system:

$$H_{\text{hyp-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j \neq k} \left[1 + \frac{\sinh^2 x}{\sinh^2(q_k - q_j)} \right]^{\frac{1}{2}}$$

• Our purpose is to derive all of Ruijsenaars' dualities by reductions of suitable finite-dimensional phase spaces. Then study new cases: systems with two types of particles, $BC(n)$ systems etc.

• Today, I describe the non-self-dual cases of the duality.

Duality from symplectic reduction: the basic idea

Start with 'big phase space', of group theoretic origin, equipped with *two* commuting families of 'canonical free Hamiltonians'.

Apply suitable *single* symplectic reduction to the big phase space and construct *two* 'natural' models of the reduced phase space.

The two families of 'free' Hamiltonians turn into interesting **many-body Hamiltonians** and **particle-position variables** in terms of *both* models. Their rôle is *interchanged* in the two models.

The natural symplectomorphism between the two models of the reduced phase space yields the 'duality symplectomorphism'.

Motivated by KKS [78], the above 'scenario' was described by Gorsky and Nekrasov in the nineties (see e.g. Fock-Gorsky-Nekrasov-Roubtsov [2000]). They focused on local questions working mostly with infinite-dimensional phase spaces and in a complex holomorphic setting. Global structure of real phase spaces is non-trivial.

Duality between hyperbolic Sutherland and rational RS

Take real Lie algebra $gl(n,\mathbb{C})$ with bilinear form $\langle X,Y\rangle := \Re\mathrm{tr}\,(XY)$, and minimal coadjoint orbit of $U(n)$: $\mathcal{O}_x := \{\xi = \mathrm{i}x(\mathbf{1}_n - vv^\dagger)\,|\, v\in\mathbb{C}^n,\ |v|^2 = n\}$. Start with the 'big phase space' (M,Ω_M):

$$M := T^*GL(n,\mathbb{C})\times\mathcal{O}_x \simeq (GL(n,\mathbb{C})\times gl(n,\mathbb{C}))\times\mathcal{O}_x = \{(g,J^R,\xi)\}.$$

Introduce matrix functions $\mathcal{L}$ and $\hat{\mathcal{L}}$ on M by

$$\mathcal{L}(g,J^R,\xi) := J^R \quad \text{and} \quad \hat{\mathcal{L}}(g,J^R,\xi) := gg^\dagger.$$

These 'unreduced Lax matrices' generate 'canonical free Hamiltonians'

$$H_k := \frac{1}{k}\Re\mathrm{tr}\,(\mathcal{L}^k), \qquad \hat{H}_{\pm k} := \pm\frac{1}{2k}\mathrm{tr}\,(\hat{\mathcal{L}}^k), \qquad k = 1,\ldots,n$$

We shall reduce by symmetry group

$$K := U(n)\times U(n),$$

where $(\eta_L,\eta_R)\in K$ acts on M by symplectomorphism

$$\Psi_{\eta_L,\eta_R} : (g,J^R,\xi)\mapsto(\eta_L g\eta_R^{-1},\eta_R J^R\eta_R^{-1},\eta_L\xi\eta_L^{-1})$$

generated by moment map

$$\Phi : M\to u(n)\oplus u(n), \qquad \Phi(g,J^R,\xi) = ((gJ^Rg^{-1})_{u(n)}+\xi, -J^R_{u(n)})$$

Use **two** models of **the** reduced phase space: $M_{\mathrm{red}} := M//_0K \equiv \Phi^{-1}(0)/K$.

First model: the Sutherland gauge slice S

Consider the Weyl chamber: $\mathcal{C} := \{\, q \in \mathbb{R}^n \,|\, q_1 > q_2 > \cdots > q_n \,\}$. $T^*\mathcal{C} \simeq \mathcal{C} \times \mathbb{R}^n = \{(q,p)\}$ has Darboux form $\Omega_{T^*\mathcal{C}} = \sum_k dp_k \wedge dq_k$. Define Hermitian matrix function L on $T^*\mathcal{C}$ by

$$L(q,p)_{jk} := p_j \delta_{jk} - \mathrm{i}(1-\delta_{jk}) \frac{x}{\sinh(q_j - q_k)}$$

L is the standard Lax matrix of the hyperbolic Sutherland model. Recall that $\mu(x) \in \mathcal{O}_{-x} \subset u(n)$ is given by $\mu(x)_{jj} = 0$ and $\mu(x)_{jk} = \mathrm{i}x$ for all $j \neq k$. Identify any $q \in \mathbb{R}^n$ with $q \simeq \mathrm{diag}(q_1, \ldots, q_n)$.

Theorem 1. *The manifold S defined by*

$$S := \{\, (e^q, L(q,p), -\mu(x)) \,|\, (q,p) \in \mathcal{C} \times \mathbb{R}^n \,\}$$

is a **global cross section** *of the K-orbits in $\Phi^{-1}(0) \subset M$.*
If $\iota_S : S \to M$ is the injection, then in terms of the coordinates q, p on S one has $\iota_S^*(\Omega_M) = \sum_k dp_k \wedge dq_k$. Thus, the symplectic manifold

$$(S, \sum_k dp_k \wedge dq_k) \simeq (T^*\mathcal{C}, \Omega_{T^*\mathcal{C}})$$

is a model of the reduced phase space.

Goes back to Olshanetsky-Perelomov [76], Kazhdan-Kostant-Sternberg [78].

Next, denote the elements of $T^*\mathcal{C} = \mathcal{C} \times \mathbb{R}^n$ as pairs $(\hat{p}, \hat{q})$.

Define (Hermitian, positive definite) matrix-function $\hat{L}$ on $T^*\mathcal{C}$ by

$$\hat{L}(\hat{p},\hat{q})_{jk} = u_j(\hat{p},\hat{q}) \left[\frac{\mathrm{i}x}{\mathrm{i}x + (\hat{p}_j - \hat{p}_k)} \right] u_k(\hat{p},\hat{q}),$$

$$u_j(\hat{p},\hat{q}) := e^{-\hat{q}_j/2} \prod_{m \neq j} \left[1 + \frac{x^2}{(\hat{p}_j - \hat{p}_m)^2} \right]^{\frac{1}{4}}, \quad j = 1, \ldots, n.$$

Then define $\mathbb{R}^n$-valued function

$$v(\hat{p},\hat{q}) := \hat{L}(\hat{p},\hat{q})^{-\frac{1}{2}} u(\hat{p},\hat{q}) \quad \text{with} \quad u = (u_1, \ldots, u_n)^T.$$

Finally, introduce the $\mathcal{O}_x$-valued function

$$\xi(\hat{p},\hat{q}) := \xi(v(\hat{p},\hat{q})) = \mathrm{i}x(\mathbf{1}_n - v(\hat{p},\hat{q}) v(\hat{p},\hat{q})^\dagger)$$

$\hat{L}$ is the standard Lax matrix of the rational Ruijsenaars-Schneider system.

Second model: the Ruijsenaars gauge slice $\widehat{S}$

Theorem 2. *The manifold $\widehat{S}$ defined by*

$$\widehat{S} := \{\, (\hat{L}(\hat{p},\hat{q})^{\frac{1}{2}}, 2\hat{p}, \xi(\hat{p},\hat{q})) \,|\, (\hat{p},\hat{q}) \in \mathcal{C} \times \mathbb{R}^n \,\}$$

is a **global cross section** *of the K-orbits in $\Phi^{-1}(0) \subset M$.*
If $\iota_{\widehat{S}} : \widehat{S} \to M$ is the injection, then in terms of the coordinates $\hat{p}$, $\hat{q}$ on $\widehat{S}$ one has $\iota_{\widehat{S}}^*(\Omega_M) = \sum_k d\hat{q}_k \wedge d\hat{p}_k$. Therefore, the symplectic manifold

$$(\widehat{S}, \sum_k d\hat{q}_k \wedge d\hat{p}_k) \simeq (T^*\mathcal{C}, \Omega_{T^*\mathcal{C}})$$

is a model of the reduced phase space.

- Theorem 2 is our main result in arXiv:0901.1983.

- At an intermediate stage of the reduction we reach $T^*\big(GL(n,\mathbb{C})/U(n)\big)$, with Riemannian symmetric space $G(n,\mathbb{C})/U(n)$. We could have started here.

Consequences

Since S and $\widehat{S}$ are two models of **the** reduced phase space $M_{\rm red}$, there exists a natural symplectomorphism between the two models:

$$(S, \sum_k dp_k \wedge dq_k) \equiv (M//_0 K, \Omega_{\rm red}) \equiv (\widehat{S}, \sum_k d\hat{q}_k \wedge d\hat{p}_k).$$

The 'free' Hamiltonians H_j and $\widehat{H}_{\pm k}$ descend to integrable reduced Hamiltonians $H_j^{\rm red}$ and $\widehat{H}_{\pm k}^{\rm red}$ on $M_{\rm red}$.

The reduced Hamiltonians take following form in terms of the 'gauge slices' $(S, \sum_k dp_k \wedge dq_k)$ and $(\widehat{S}, \sum_k d\hat{q}_k \wedge d\hat{p}_k)$:

on S : $\quad H_j^{\rm red} = \frac{1}{j}{\rm tr}\,(L^j)\,, \qquad \widehat{H}_{\pm k}^{\rm red} = \pm\frac{1}{2k}\sum_{i=1}^n (e^{2q_i})^{\pm k}$

on $\widehat{S}$: $\quad H_j^{\rm red} = \frac{1}{j}\sum_{i=1}^n (2\hat{p}_i)^j\,, \qquad \widehat{H}_{\pm k}^{\rm red} = \pm\frac{1}{2k}{\rm tr}\,(\hat{L}^{\pm k})$

The natural symplectomorphism is Ruijsenaars' duality map.

Trigonometric Sutherland – following KKS [78]

Consider cotangent bundle $T^*U(n)$ of $U(n)$ (in right-trivialization):

$$T^*U(n) = \{(g, J_L) \,|\, g \in U(n),\; J_L \in u(n)^* \simeq u(n)\}$$

It carries the natural symplectic form

$$\Omega(g, J_L) = d\,\mathrm{tr}\,(J_L dg g^{-1})$$

and two sets of 'canonical free Hamiltonians' $\{h_k\}$ and $\{\widehat{h}_{\pm k}\}$

$$h_k(g, J_L) := \mathrm{tr}\,(\mathrm{i}J_L)^k, \quad \widehat{h}_k(g, J_L) := \Re\mathrm{tr}\,(g^k), \quad \widehat{h}_{-k}(g, J_L) := \Im\mathrm{tr}\,(g^k)$$

- One can write down their Hamiltonian flows explicitly.

- They are invariant under the adjoint action of $U(n)$ on $T^*U(n)$:

$$\eta \triangleright (g, J_L) = (\eta g \eta^{-1}, \eta J_L \eta^{-1}) \qquad \forall \eta \in U(n),$$

generated by the moment map $J : T^*U(n) \to u(n)^*$ given by

$$J(g, J_L) = J_L + J_R \quad \text{with} \quad J_R(g, J_L) := -g^{-1} J_L g.$$

J is sum of moment maps generating left/right multiplication.

KKS [78] found that the moment map constraint $J = \mu(x)$ produces the trigonometric Sutherland system from the Hamiltonian system describing the free particle on $U(n)$: $(T^*U(n), \Omega, h_2)$. The Hamiltonians $\{h_k\}$ give action variables of Sutherland system (and $\{\widehat{h}_{\pm k}\}$ become in effect the Sutherland particle-positions).

It can be shown that using another model of the reduced phase space $\{\widehat{h}_{\pm k}\}$ yield the commuting Hamiltonians of the Ruijsenaars dual of the Sutherland system (and $\{h_k\}$ become in effect the dual particle positions).

Recently in 1005.4531 [math-ph] (V. Ayadi and L.F.: Trigonometric Sutherland systems and their Ruijsenaars duals from symplectic reduction), we considered covering homomorphisms

$$G_2 := \mathbb{R} \times SU(n) \longrightarrow G_1 := U(1) \times SU(n) \longrightarrow G := U(n)$$

and 'KKS reductions' of the 3 cotangent bundles by the effective symmetry group

$$\bar{G} := G/\mathbb{Z}_G \simeq G_1/\mathbb{Z}_{G_1} \simeq G_2/\mathbb{Z}_{G_2}.$$

This 'explained' the web of dualities and coverings due to Ruijsenaars [95]:

$$\begin{array}{ccc}
T^*\mathbb{R} \times T^*SQ(n) & \xrightarrow{\mathrm{id}_2 \times \mathcal{R}_0} & T^*\mathbb{R} \times \mathbb{C}^{n-1} \\
\downarrow \psi_2^{\mathrm{I}} & & \downarrow \psi_2^{\mathrm{II}} \\
T^*U(1) \times T^*SQ(n) & \xrightarrow{\mathrm{id}_1 \times \mathcal{R}_0} & T^*U(1) \times \mathbb{C}^{n-1} \\
\downarrow \psi_1^{\mathrm{I}} & & \downarrow \psi_1^{\mathrm{II}} \\
P = T^*Q(n) & \xrightarrow{\mathcal{R}} & \widehat{P}_c = \mathbb{C}^{n-1} \times \mathbb{C}^{\times}
\end{array}$$

$Q(n) = \mathbb{T}_n^0/S_n$ is the configuration space of n indistinguishable non-colliding point particles moving on the circle and $SQ(n)$ belongs to the relative motion of n distinguishable particles. On the right-side the corresponding completed dual phase spaces appear and the vertical maps are coverings.

As our final example, we deal with the standard trigo RS system, whose phase space is $P := T^*Q(n)$. Here, $Q(n) := \mathbb{T}_n^0/S_n$ with $\mathbb{T}_n^0$ being the regular part of the maximal torus $\mathbb{T}_n < U(n)$.
The corresponding Lax matrix L and symplectic form ω are:

$$L_{jk}(q,p) = \frac{e^{p_k}\sinh(-x)}{\sinh(\mathrm{i}q_j - \mathrm{i}q_k - x)} \prod_{m\neq j}\left[1 + \frac{\sinh^2 x}{\sin^2(q_j - q_m)}\right]^{\frac{1}{4}} \prod_{m\neq k}\left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_m)}\right]^{\frac{1}{4}}$$

$$\omega = \sum_k dp_k \wedge dq_k, \qquad p_k \in \mathbb{R}, \qquad 0 \leq q_k < \pi, \quad q_1 > q_2 > \cdots > q_n$$

The dual system can be *locally* characterized by

$$\hat{L}_{jk}(e^{\mathrm{i}\hat{q}},\hat{p}) = \frac{e^{\mathrm{i}\hat{q}_k}\sinh(-x)}{\sinh(\hat{p}_j - \hat{p}_k - x)} \prod_{m\neq j}\left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_j - \hat{p}_m)}\right]^{\frac{1}{4}} \prod_{m\neq k}\left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_m)}\right]^{\frac{1}{4}}$$

$\hat{p} = \mathrm{diag}(\hat{p}_1,\dots,\hat{p}_n) \in \mathfrak{C}_x := \{\hat{p}\,|\,\hat{p}_j - \hat{p}_{j+1} > |x|, \ \ j = 1,\dots,(n-1)\}$
$e^{\mathrm{i}\hat{q}} \in \mathbb{T}_n$ with $\hat{q} = \mathrm{diag}(\hat{q}_1,\dots,\hat{q}_n)$. Dual phase space $\hat{P} = \mathbb{T}_n \times \mathfrak{C}_x$ is open submanifold of cotangent bundle of $\mathbb{T}_n$, with $\hat{\omega} = d\hat{p}_k \wedge d\hat{q}_k$.

- The commuting flows associated with $\hat{L}$ are **not** complete on $\hat{P}$.
- $\hat{P}$ is symplectomorpic **(only)** to a dense, open submanifold of P.

Hence $\hat{P}$ needs to be extended, as performed by Ruijsenaars [95].

Poisson-Lie analogue of Kazhdan-Kostant-Sternberg reduction

According to Semenov-Tian-Shansky [85] and Lu-Weinstein [90]:

- P-L analogue of $T^*U(n)$ is Heisenberg double of Poisson $U(n)$.

The Heisenberg double of $U(n)$ is the *real* manifold $GL(n,\mathbb{C})$.

Every $K \in GL(n,\mathbb{C})$ admits two Iwasawa decompositions:

$$K = b_L g_R^{-1} \quad \text{and} \quad K = g_L b_R^{-1} \quad \text{with} \quad g_{L,R} \in U(n),\ b_{L,R} \in B$$

B: group of upper triangular matrices with positive diagonal entries

Define maps $\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ and $\Xi_{L,R} : GL(n,\mathbb{C}) \to U(n)$

$$\Lambda_{L,R}(K) := b_{L,R} \quad \text{and} \quad \Xi_{L,R}(K) := g_{L,R}$$

$GL(n,\mathbb{C})$ has natural symplectic form (Alekseev-Malkin [94])

$$\omega_+ = \frac{1}{2}\Im\mathrm{tr}\,(d\Lambda_L\Lambda_L^{-1} \wedge d\Xi_L\Xi_L^{-1}) + \frac{1}{2}\Im\mathrm{tr}\,(d\Lambda_R\Lambda_R^{-1} \wedge d\Xi_R\Xi_R^{-1})$$

Commuting Hamiltonians from dual P-L groups

Iwasawa maps $\Xi_{L,R} : GL(n,\mathbb{C}) \to U(n)$ and $\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ are **Poisson maps** if $U(n)$ and B are equipped with their standard Poisson structures. In fact, the Poisson bracket $\{\ ,\ \}_+$ defined by ω_+ closes on

$$\Xi^*_{L,R} C^\infty(U(n)) \quad \text{and on} \quad \Lambda^*_{L,R} C^\infty(B)$$

Induced Poisson bracket on $U(n)$ is standard Sklyanin bracket

[defined by Drinfeld-Jimbo r-matrix, $R^{\mathrm{i}} \in \mathrm{End}(u(n))$, $R^{\mathrm{i}}(X) = \pi_{u(n)}(-\mathrm{i}X)$]

$C^\infty(U(n))^{U(n)}$: the adjoint (conjugation) invariant functions

$C^\infty(B)^c \equiv C^\infty(B)^{U(n)}$: the center of the Poisson bracket on $C^\infty(B)$ provided by the dressing invariants

$$\Lambda^*_L C^\infty(B)^c = \Lambda^*_R C^\infty(B)^c \quad \text{and} \quad \Xi^*_R C^\infty(U(n))^{U(n)}$$

form **Abelian subalgebras** in $C^\infty(GL(n,\mathbb{C}))$ w.r.t. $\{\ ,\ \}_+$

The 'canonical free flows'

- First, flow of Hamiltonian $H = f \circ \Lambda_R$ with $f \in \mathbf{C^\infty(B)^c}$ is

$$K(t) = g_L(t) b_R^{-1}(t) = g_L(0) \exp\left[-t d^R f(b_R(0))\right] b_R^{-1}(0)$$

In other words, $b_R(t) = b_R(0)$ and $g_L(t) = g_L(0) \exp\left[-t d^R f(b_R(0))\right]$

Equivalently, $b_L(t) = b_L(0)$ and $g_R(t) = \exp\left[-t d^L f(b_L(0))\right] g_R(0)$

- Second, the flow of $\hat{H} = \phi \circ \Xi_R$ with $\phi \in \mathbf{C^\infty(U(n))^{U(n)}}$ reads

$$g_R(t) = \gamma(t) g_R(0) \gamma(t)^{-1}, \qquad b_L(t) = b_L(0)\beta(t)$$

with $\gamma(t) \in U(n)$, $\beta(t) \in B$ defined by $e^{\mathrm{i}t\mathbf{D}\phi(g_R(0))} = \beta(t)\gamma(t)$. Also

$$K(t)K^\dagger(t) = b_L(t) b_L(t)^\dagger = b_L(0) e^{2\mathrm{i}t\mathbf{D}\phi(g_R(0))} b_L(0)^\dagger$$

Solutions are obtained by Gram-Schmidt orthogonalization.

Quasi-adjoint symmetry

Following Lu [90]:
Poisson map from phase space into P-L group B is called (equivariant) *P-L moment map*. Every such map generates infinitesimal Poisson action of $U(n)$

$\Lambda_{L,R} : GL(n,\mathbb{C}) \to B$ moment maps generating left/right multiplications by $U(n)$.
The product $\Lambda := \Lambda_L \Lambda_R : GL(n,\mathbb{C}) \to B$ is also P-L moment map.
Λ generates infinitesimal 'quasi-adjoint' action of $U(n)$.

Concretely, for any $Y \in u(n)$ define vector field $\tilde{Y}$ on $GL(n,\mathbb{C})$ by

$$\mathcal{L}_{\tilde{Y}} f := \Im \mathrm{tr}\,(Y\{f,\Lambda\}_+ \Lambda^{-1}), \qquad \forall f \in C^\infty(GL(n,\mathbb{C}))$$

Integration of infinitesimal action yields $U(n)$ action on $GL(n,\mathbb{C})$:

$$\eta \triangleright K := \eta K \Xi_R(\eta \Lambda_L(K)), \qquad \eta \in U(n), \quad K \in GL(n,\mathbb{C})$$

Now can reduce $(GL(n,\mathbb{C}),\omega_+)$ by choosing $\nu \in B$ and imposing

moment map constraint: $\Lambda(K) = \nu, \quad K \in GL(n,\mathbb{C}).$

'Canonical free Hamiltonians' are invariant under the quasi-adjoint action of $U(n)$; thus can be reduced simultaneously. **In this way we obtained 'trigonometric Ruijsenaars duality' from P-L duality.**

‘Unreduced Lax matrices’

generators of $C^\infty(B)^c$: $\quad f_k(b) := \frac{1}{2k}\text{tr}\,(bb^\dagger)^k \quad \forall k \in \mathbb{Z}^*$
/$C^\infty(B)^c = C^\infty(B)^{U(n)}$ – dressing invariants/

$$\text{generators of } C^\infty(U(n))^{U(n)}: \quad \phi_k(g) := \frac{1}{2k}\text{tr}\,(g^k + g^{-k})$$

$$\phi_{-k}(g) := \tfrac{1}{2k\mathrm{i}}\text{tr}\,(g^k - g^{-k}) \quad \forall k \in \mathbb{Z}_+$$

Canonical Hamiltonians $H_k := f_k \circ \Lambda_R$ and $\hat{H}_k := \phi_k \circ \Xi_R$ are **spectral invariants** of matrix functions $\mathcal{L}$ and $\hat{\mathcal{L}}$ defined on the double by

$$\mathcal{L} := \Lambda_R \Lambda_R^\dagger \quad \text{and} \quad \hat{\mathcal{L}} := \Xi_R$$

We call $\mathcal{L}$ and $\hat{\mathcal{L}}$ unreduced Lax matrices.

The quasi-adjoint action operates on the ‘unreduced Lax matrices’ $\mathcal{L}$ and $\hat{\mathcal{L}}$ by similarity transformations. Hence $\mathcal{L}$ and $\hat{\mathcal{L}}$ yield Lax matrices for reduced systems obtained from $\{H_k\}$ and from $\{\hat{H}_k\}$.
We proved: $\mathcal{L}$ and $\hat{\mathcal{L}}$ descend to the trigo RS Lax matrices L and $\hat{L}$.

Definition of the reduction

- First, fix value of moment map Λ to some constant $\nu \in B$.
- Second, factor level set $\Lambda^{-1}(\nu)$ by isotropy group G_ν of ν.

The crux is the choice $\nu := \nu(x)$: with $x \neq 0$ real parameter

$$\nu(x)_{kk} = 1, \quad \forall k, \quad \nu(x)_{kl} = (1 - e^{-2x})e^{(l-k)x}, \quad \forall k < l$$

Useful relation: $\nu(x)\nu(x)^\dagger = e^{-2x}\left[\mathbf{1}_n + \frac{e^{2nx} - 1}{n} v(x)v(x)^\dagger\right]$

with vector $v(x) \in \mathbb{R}^n$ defined by $v_k(x) = \sqrt{\frac{n(e^{2x}-1)}{1-e^{-2nx}}}e^{-kx}$

$F_{\nu(x)} := \Lambda^{-1}(\nu(x))$: **embedded** submanifold of $GL(n, \mathbb{C})$
$G_{v(x)} < U(n)$: isotropy group of $v(x)$ – **acts freely** on $F_{\nu(x)}$

Central $U(1) < U(n)$ acts trivially. $G_{v(x)} < G_{\nu(x)}$ isomorphic to $G_{\nu(x)}/U(1)$.
Reduced phase space is **smooth manifold** $F_{\nu(x)}/G_{v(x)}$.

We exhibit two models, which will be identified with (P, ω) and with the natural completion of $(\hat{P}, \hat{\omega})$, explaining this case of the duality.

Important features of the reduced system

Consider natural embedding $\mathcal{E}$ and projection π

$$\mathcal{E} : F_{\nu(x)} \to D \equiv GL(n, \mathbb{C}), \quad \pi : F_{\nu(x)} \to F_{\nu(x)}/G_{v(x)} \equiv D_{\text{red}}$$

$(D_{\text{red}}, \omega_{\text{red}})$ is symplectic manifold characterized by $\mathcal{E}^*\omega_+ = \pi^*\omega_{\text{red}}$

$(D_{\text{red}}, \omega_{\text{red}})$ carries reduced canonical Hamiltonians defined by

$$\pi^* H_k^{\text{red}} = \mathcal{E}^* H_k, \qquad \pi^* \widehat{H}_k^{\text{red}} = \mathcal{E}^* \widehat{H}_k$$

$\{H_k^{\text{red}}\}$ and $\{\widehat{H}_k^{\text{red}}\}$ form two **Abelian** algebras. Induce **complete flows** on D_{red}: obvious projections of 'canonical free flows'.

The aim is to exhibit concrete models of the reduced phase space.

Preparation for describing the first model

Consider

$$T^*\mathbb{T}_n^0 \simeq \mathbb{T}_n^0 \times \mathbb{R}^n = \{(e^{2\mathrm{i}q}, p)\}, \qquad \Omega_{T^*\mathbb{T}_n^0} \equiv \sum_{k=1}^n dp_k \wedge dq_k$$

and the projection $\pi_1 : T^*\mathbb{T}_n^0 \to (T^*\mathbb{T}_n^0)/S_n \equiv T^*(\mathbb{T}_n^0/S_n) \equiv T^*Q(n)$, for which $\pi_1^*(\Omega_{T^*Q(n)}) = \Omega_{T^*\mathbb{T}_n^0}$. That is, consider S_n-covering of phase space $P = T^*Q(n)$.

Define the smooth map $\tilde{\mathcal{I}} : T^*\mathbb{T}_n^0 \to GL(n,\mathbb{C})$ by the following explicit formula:

$$\tilde{\mathcal{I}}(e^{2\mathrm{i}q}, p)_{kk} = e^{-p_k/2 - 2\mathrm{i}q_k} \prod_{m<k} \left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_m)}\right]^{-\frac{1}{4}} \prod_{m>k} \left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_m)}\right]^{\frac{1}{4}}$$

$$\tilde{\mathcal{I}}(e^{2\mathrm{i}q}, p)_{kl} = 0, \quad k > l, \qquad \tilde{\mathcal{I}}(e^{2\mathrm{i}q}, p)_{kl} = \tilde{\mathcal{I}}(e^{2\mathrm{i}q}, p)_{ll} \prod_{m=1}^{l-k} \frac{e^x e^{2\mathrm{i}q_l} - e^{-x} e^{2\mathrm{i}q_{k+m}}}{e^{2\mathrm{i}q_l} - e^{2\mathrm{i}q_{k+m-1}}} \quad k < l$$

Claim: the image of $T^*\mathbb{T}_n^0$ by $\tilde{\mathcal{I}}$ is a symplectic submanifold $\tilde{S} \subset F_{\nu(x)} \subset GL(n,\mathbb{C})$. $(\tilde{S}, \omega_+|_{\tilde{S}})$ and $T^*\mathbb{T}_n^0$ are symplectomorphic by $\tilde{\mathcal{I}}$, and furnish symplectic covering spaces of the reduced phase space.

The first model of the reduced phase space

The map $\tilde{\mathcal{I}} : T^*\mathbb{T}_n^0 \to D$ is injective, its image lies in $F_{\nu(x)}$, and it verifies

$$\tilde{\mathcal{I}}^* \omega_+ = \Omega_{T^*\mathbb{T}_n^0}.$$

$\tilde{\mathcal{I}}$ descends to a diffeomorphism $\mathcal{I} : T^*Q(n) \to F_{\nu(x)}/G_{v(x)}$ defined by the equality

$$\mathcal{I} \circ \pi_1 = \pi \circ \tilde{\mathcal{I}},$$

and $\mathcal{I}$ satisfies $\mathcal{I}^*\omega_{\text{red}} = \Omega_{T^*Q(n)}$, where $\pi : F_{\nu(x)} \to F_{\nu(x)}/G_{v(x)} \equiv D_{\text{red}}$ is projection.

Thus $(P, \omega) \equiv (T^*Q(n), \Omega_{T^*Q(n)})$ is a model of reduced phase space $(D_{\text{red}}, \omega_{\text{red}})$.

With $\tilde{S} \subset \Lambda^{-1}(\nu(x)) \equiv F_{\nu(x)}$, the situation is summarized by the diagram:

$$\begin{array}{ccc} T^*\mathbb{T}_n^0 & \xrightarrow{\tilde{\mathcal{I}}} & \tilde{S} \subset F_{\nu(x)} \\ \pi_1 \downarrow & & \downarrow \pi \\ T^*Q(n) & \xrightarrow{\mathcal{I}} & \tilde{S}/S_n \simeq D_{\text{red}} \end{array} \qquad \text{with induced } S_n\text{-action on } \tilde{S}.$$

The composition $\mathcal{L} \circ \tilde{\mathcal{I}}$ gives (up to inessential similarity transformation) the Lax matrix L of the original Ruijsenaars-Schneider system, where L is regarded as a function on the covering space $T^*\mathbb{T}_n^0$ of $P = T^*Q(n)$.

Hence trigo RS system (P, ω, L) is reduction of 'free' system $(D, \omega_+, \mathcal{L})$.

Preparations for the second model

Recall (incomplete) dual phase space, $\hat{P} = \mathbb{T}_n \times \mathfrak{C}_x = \{(e^{\mathrm{i}\hat{q}}, \hat{p})\}$ with $\hat{\omega} = d\hat{p}_k \wedge d\hat{q}_k$.

Consider $\hat{P}_c := \mathbb{C}^{n-1} \times \mathbb{C}^\times$ with the symplectic form

$$\hat{\omega}_c := \frac{\mathrm{i}dZ \wedge d\bar{Z}}{2\bar{Z}Z} + \mathrm{sign}(x) \sum_{j=1}^{n-1} \mathrm{i}dz_j \wedge d\bar{z}_j, \qquad Z \in \mathbb{C}^\times, \quad z \in \mathbb{C}^{n-1}.$$

Define the smooth injective map $\mathcal{Z}_x : \hat{P} \to \hat{P}_c$ by

$$z_j(x, \hat{q}, \hat{p}) = (\hat{p}_j - \hat{p}_{j+1} - |x|)^{\frac{1}{2}} \prod_{k=j+1}^{n} e^{-\mathrm{i}\hat{q}_k}, \quad Z(x, \hat{q}, \hat{p}) = e^{-\hat{p}_1} \prod_{k=1}^{n} e^{-\mathrm{i}\hat{q}_k}, \quad x > 0,$$

$$z_j(x, \hat{q}, \hat{p}) = (\hat{p}_j - \hat{p}_{j+1} - |x|)^{\frac{1}{2}} \prod_{k=1}^{j} e^{-\mathrm{i}\hat{q}_k}, \quad Z(x, \hat{q}, \hat{p}) = e^{-\hat{p}_n} \prod_{k=1}^{n} e^{-\mathrm{i}\hat{q}_k}, \quad x < 0.$$

$\mathcal{Z}_x$ is a symplectic embedding of $(\hat{P}, \hat{\omega})$ into $(\hat{P}_c, \hat{\omega}_c)$, $\mathcal{Z}_x^* \hat{\omega}_c = \hat{\omega}$.

The $\mathcal{Z}_x$-image $\hat{P}_c^0 := \mathcal{Z}_x(\hat{P}) \subset \hat{P}_c$ is dense open submanifold.
$\hat{P}_c \setminus \mathcal{Z}_x(\hat{P})$ consists of the points for which some z_j $(j = 1, ..., n-1)$ vanishes.

With $\widehat{p} := \mathrm{diag}(\widehat{p}_1, \ldots, \widehat{p}_n)$, define $\mathrm{O}(n,\mathbb{R})$-valued function θ on the closure of $\mathfrak{C}_x$:

$$\theta(x,\widehat{p})_{jk} := \frac{\sinh(x)}{\sinh(\widehat{p}_k - \widehat{p}_j)} \prod_{m\neq j,k} \left[\frac{\sinh(\widehat{p}_j - \widehat{p}_m - x)\sinh(\widehat{p}_k - \widehat{p}_m + x)}{\sinh(\widehat{p}_j - \widehat{p}_m)\sinh(\widehat{p}_k - \widehat{p}_m)}\right]^{\frac{1}{2}}, \text{ if } j \neq k,$$

$$\theta(x,\widehat{p})_{jj} := \prod_{m\neq j} \left[\frac{\sinh(\widehat{p}_j - \widehat{p}_m - x)\sinh(\widehat{p}_j - \widehat{p}_m + x)}{\sinh^2(\widehat{p}_j - \widehat{p}_m)}\right]^{\frac{1}{2}}.$$

We also use $\mathrm{O}(n,\mathbb{R})$-valued functions $\kappa_L(x)$ and $\zeta(x,\widehat{p})$ and the diffeomorphism $\aleph : \mathbb{T}_n \to \mathbb{T}_n$ provided by

$$\aleph(x,\tau)_j := \prod_{k=j}^{n} \tau_k^{-1}, \quad x > 0, \qquad \aleph(x,\tau)_j := \prod_{k=1}^{j} \tau_k^{-1}, \quad x < 0,$$

and notation

$$\tau_{(x)} := \mathrm{diag}(\tau_2, \ldots, \tau_n, 1) \quad \text{if} \quad x > 0, \qquad \tau_{(x)} := \mathrm{diag}(1, \tau_1, \ldots, \tau_{n-1}) \quad \text{if} \quad x < 0.$$

Finally, define smooth, injective map $k_x : \widehat{P} \to F_{\nu(x)}$ by explicit formula

$$k_x(e^{\mathrm{i}\widehat{q}}, \widehat{p}) := \left(\kappa_L(x)\aleph(x, e^{\mathrm{i}\widehat{q}})_{(x)}\zeta(x,\widehat{p})^{-1}\right) \triangleright \left(\theta(x,\widehat{p})e^{\mathrm{i}\widehat{q}}e^{\widehat{p}}\right)^{-1}$$

For full details and the derivation of this formula, see our paper arXiv:0906.4198 [math-ph].

The final result

• $\pi \circ k_x : \hat{P} \to D_{\text{red}}$ gives symplectic diffeomorphism onto open dense submanifold D^0_{red} of reduced phase space.

• $\hat{\mathcal{L}} \circ k_x$ gives (up to inessential similarity transformation) the dual Lax matrix $\hat{L}$.

• Thus $(\hat{P}, \hat{\omega}, \hat{L})$ represents the restriction on D^0_{red} of the reduction of the 'free' system $(D, \omega_+, \hat{\mathcal{L}})$.

• **The map $k_x \circ \mathcal{Z}_x^{-1} : \hat{P}^0_c \to F_{\nu(x)}$ extends uniquely to a smooth injective map $\hat{\mathcal{I}} : \hat{P}_c \to F_{\nu(x)}$ such that $\pi \circ \hat{\mathcal{I}} : \hat{P}_c \to D_{\text{red}}$ is a symplectic diffeomorphism. Therefore, $(\hat{P}_c, \hat{\omega}_c)$ is a model of the full reduced phase space.**

Ruijsenaars' restricted and global duality (action-angle) maps, $\mathcal{R}^0$ and $\mathcal{R}$, are obtained geometrically:

$$\begin{array}{ccccc} P^0 & \xrightarrow{\text{id}} & P^0 & \xrightarrow{\mathcal{I}^0} & F^0_{\nu(x)}/G_{v(x)} \\ \mathcal{R}^0 \downarrow & & \mathcal{R}^0_c \downarrow & & \downarrow \text{id} \\ \hat{P} & \xrightarrow{\mathcal{Z}_x} & \hat{P}^0_c & \xrightarrow{\pi\circ\hat{\mathcal{I}}^0} & F^0_{\nu(x)}/G_{v(x)} \end{array} \quad \text{and} \quad \begin{array}{ccc} P & \xrightarrow{\mathcal{I}} & F_{\nu(x)}/G_{v(x)} \\ \mathcal{R} \downarrow & & \downarrow \text{id} \\ \hat{P}_c & \xrightarrow{\pi\circ\hat{\mathcal{I}}} & F_{\nu(x)}/G_{v(x)} \end{array}$$

All $K \in F_{\nu(x)}$ satisfy $-\frac{1}{2}\log(KK^\dagger) \in \bar{\mathfrak{C}}_x$. Dense submanifold $F^0_{\nu(x)}$ is characterized by condition $-\frac{1}{2}\log(KK^\dagger) \in \mathfrak{C}_x$. $\hat{P}$ and P^0 are two models of $D^0_{\text{red}} \equiv F^0_{\nu(x)}/G_{v(x)}$.

Concluding remarks

Presented group theoretical method that yields many-body systems together with geometric interpretation of their duality relations.

Technically simplifies parts of original work of Ruijsenaars [88,95].

Main advantage:

Completion of local phase spaces and duality symplectomorphisms result automatically, once the correct starting point is 'guessed'.

References: arXiv:0809.1509, 0901.1983, 0906.4198, 1005.4531 [math-ph]

Problems under investigation and plans for the future:

- Study compactified, hyperbolic and elliptic RS systems.

- Explore reduced systems at arbitrary moment map value.

- Quantum Hamiltonian reduction ($\sim$ works on special functions)

Etingof-Kirillov [94], Noumi [96]: Q.G. interpretation of Macdonald polynomials

- Connections to bispectrality and to separation of variables.

- Derive $BC(n)$ (van Diejen) systems in analogous manner.