Compact forms of the Ruijsenaars-Schneider system

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Integrable systems of Calogero (Moser, Sutherland, Ruijsenaars-Schneider, Toda) type describe point "particles" moving on the line or on circle.

These systems are closely connected to soliton theory, e.g. to the KdV and sine-Gordon models, as well as to Yang-Mills and Chern-Simons field theories, and have links to important areas of mathematics.

They enjoy intriguing "duality relations".

By definition, two integrable many-body systems are dual to each other if action variables of system (i) are particle positions of system (ii), and vice versa. Underlying phase spaces are symplectomorphic.

A special case of duality is self-duality, where the leading Hamiltonians of the two systems have the same form.

The simplest self-dual system: $H_{\mathrm{Cal}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k}\frac{x^2}{(q_k-q_j)^2}$

Symplectic reduction: Consider phase space $T^*\mathrm{i}u(n) \simeq \mathrm{i}u(n)\times \mathrm{i}u(n) := \{(Q,P)\}$ with two families of 'free' Hamiltonians $\{\mathrm{tr}\,(Q^k)\}$ and $\{\mathrm{tr}\,(P^k)\}$. Reduce by the adjoint action of $U(n)$ using the moment map constraint

$$[Q,P] = \mu(x) := \mathrm{i}x\sum_{j\neq k} E_{j,k}$$

This yields the self-dual Calogero system (OP [76], KKS [78]):

gauge slice (i): $Q = q := \mathrm{diag}(q_1,\dots,q_n)$, $q_1 > \cdots > q_n$, with $p := \mathrm{diag}(p_1,\dots,p_n)$

$$P = p + \mathrm{i}x\sum_{j\neq k}\frac{E_{jk}}{q_j-q_k} \equiv L_{\mathrm{Cal}}(q,p) \quad \text{Lax matrix,} \quad \mathrm{tr}\,(dP\wedge dQ) = \sum_{k=1}^{n} dp_k\wedge dq_k$$

gauge slice (ii): $P = \hat{p} := \mathrm{diag}(\hat{p}_1,\dots,\hat{p}_n)$, $\hat{p}_1 > \cdots > \hat{p}_n$, with $\hat{q} := \mathrm{diag}(\hat{q}_1,\dots,\hat{q}_n)$

$$Q = -L_{\mathrm{Cal}}(\hat{p},\hat{q}) \quad \text{dual Lax matrix,} \quad \mathrm{tr}\,(dP\wedge dQ) = \sum_{k=1}^{n} d\hat{q}_k\wedge d\hat{p}_k.$$

The alternative gauge slices give two models of the reduced phase space. Their natural symplectomorphism is the self-duality map.

For a recent application, see T.F. Görbe: A simple proof of Sklyanin's formula for canonical spectral coordinates of the rational Calogero-Moser system, SIGMA 12 (2016), 027

Further self-dual systems

Hyperbolic Ruijsenaars-Schneider system:

$$H_{\text{hyp-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j\neq k} \left[1 + \frac{\sinh^2 x}{\sinh^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

Its self-duality was shown by Ruijsenaars in 1988.

Compact(ified) trigonometric RS ($\mathrm{III_b}$) system, locally given by

$$H_{\text{compact-RS}} = \sum_{k=1}^{n} (\cos p_k) \sqrt{\prod_{j\neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)}\right]}$$

Ruijsenaars (1995) studied the latter system **assuming** $0 < x < \pi/n$. He proved that (after going to the 'center of mass frame') the naive phase space (corresponding to 'particles' on the circle located at $e^{\mathrm{i}2q_k}$ subject to $q_{k+1} - q_k > x$, $\forall k = 1, \dots, n$) can be compactified to $\mathbb{CP}^{n-1}$. Then the flows are complete and the system is self-dual. He also noted that similar compactification works for the elliptic RS system as well.

My talk 2012@CH: *The self-duality map as a mapping class symplectomorphism*

Plan of the talk

- I. Derivation of compact trigonometric RS systems by reduction.

- II. Direct construction of the resulting systems of type (i).

- III. Direct construction works in the elliptic case as well:

$$H_{\mathsf{elliptic-RS}} = \sum_{k=1}^{n} (\cos p_k) \sqrt{\prod_{j \neq k}^{n} \left[\mathsf{s}(x)^2 \Big(\wp(x) - \wp(q_j - q_k) \Big) \right]}\,.$$

- IV. Conclusion, and remarks on related results.

Based on joint works with T. Kluck (I.) and T.F. Görbe (II.-III.)

Reduction approach to compact trigonometric systems

For any reductive Lie group G, one can reduce the 'phase space'

$$G \times G = \{(A, B)\} \text{ by imposing constraint } ABA^{-1}B^{-1} = \mu_0$$

using any constant μ_0 and taking quotient by gauge transformations

$$(A, B) \longrightarrow (gAg^{-1}, gBg^{-1}), \;\; g \in G \text{ with } g\mu_0 g^{-1} = \mu_0.$$

Reduced phase space is the moduli space of flat G-connections on the torus with a hole, such that the holonomy around the hole is constrained to the conjugacy class of μ_0. The matrices A and B are the holonomies along the standard cycles on the torus. **Their invariant functions generate two Abelian Poisson algebras.**

The mapping class group of the "one-holed torus" – $SL(2, \mathbb{Z})$ – acts symplectically on the reduced phase space.

The idea to interpret trigonometric RS systems in terms of moduli space is due to Gorsky-Nekrasov and Fock-Rosly (mid ninenties).

Self-dual compact forms of the trigonometric RS system

Consider $G := SU(n)$ and equip the double $G \times G = \{(A, B)\}$ with the 2-form
$\omega := \big(\langle A^{-1}dA \overset{\wedge}{,} dBB^{-1}\rangle + \langle dAA^{-1} \overset{\wedge}{,} B^{-1}dB\rangle - \langle (AB)^{-1}d(AB) \overset{\wedge}{,} (BA)^{-1}d(BA)\rangle\big)$.
The 2-form, the moment map $\mu \colon (A, B) \mapsto ABA^{-1}B^{-1}$, and the action of G by componentwise conjugation makes $G \times G$ a quasi-Hamiltonian space (Alekseev-Malkin-Meinrenken, 1998).

The reduced phase space $P(\mu_0) := \mu^{-1}(\mu_0)/G_{\mu_0}$ is symplectic.

The class functions of G, applied to either components A or B in the pair $(A, B) \in G \times G$, descend to two Abelian Poisson algebras on $P(\mu_0)$.

Earlier with C. Klimcik, analyzed this quasi-Hamiltonian reduction taking

$$\mu_0 := \mu_0(x) := \mathrm{diag}\left(e^{2\mathrm{i}x}, \dots, e^{2\mathrm{i}x}, e^{-2\mathrm{i}(n-1)x}\right)$$

with $0 < x < \pi/n$. More recently with T. Kluck, studied **general case** $0 < x < \pi$.

First result: this construction always gives a self-dual integrable system on the compact, connected, smooth reduced phase space $P(\mu_0(x))$ of dimension $2(n-1)$.

Second result: On a dense open submanifold of $P(\mu_0(x))$ the "main Hamiltonian" coming from $\Re(\mathrm{tr}\,(A))$ takes the RS form of $\mathrm{III_b}$ type:

$$H_{\mathrm{compact-RS}} = \sum_{k=1}^{n} (\cos p_k) \sqrt{\prod_{j \neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)}\right]}$$

This describes n "particles" moving on the circle. Domain of "position variables" is the same as domain of "action variables" and depends on value of x.

Two types of compact RS systems

The analysis requires finding the spectra of B for all (A, B) in the constraint surface $\mu^{-1}(\mu_0(x))$, where $ABA^{-1}B^{-1} = \mu_0(x)$. $/e^{2\mathrm{i}xm} \neq 1$ for all $m = 1, 2, \ldots, n/$

In principle, two qualitatively different types of cases can occur:

- Type (i): the constraint surface satisfies $\mu^{-1}(\mu_0(x)) \subset G_{\mathrm{reg}} \times G_{\mathrm{reg}}$.
- Type (ii): the relation $\mu^{-1}(\mu_0(x)) \subset G_{\mathrm{reg}} \times G_{\mathrm{reg}}$ does not hold.

The reduced phase space $P(\mu_0(x))$ is naturally a Hamiltonian toric manifold if and only if $\mu^{-1}(\mu_0(x)) \subset G_{\mathrm{reg}} \times G_{\mathrm{reg}}$, i.e., in the type (i) cases. In other words, one obtains $(n-1)$ globally smooth, independent action variables generating an effective torus action.

Indeed, in the type (i) cases certain "spectral functions" on G that are smooth on G_{reg} but only continuous at G_{sing} descend to smooth action variables and position variables when applied to A and B with $(A, B) \in \mu^{-1}(\mu_0(x))$.

In the type (ii) cases the particles can collide and the action variables become non-differentiable at singular points, where the $(n-1)$ commuting smooth Hamiltonians lose their independence.

Our main result: We found the complete classification of the parameter $0 < x < \pi$ according to type (i) and type (ii) cases.

Classification of the coupling parameter

Main Theorem of [L.F.- T. Kluck]:
The type (i) cases are precisely those for which the coupling parameter $0 < x < \pi$ (subject to $e^{2\mathrm{i}xm} \neq 1$ for all $m = 1, 2, \dots, n$) belongs to a punctured interval of the form

$$\pi\left(\frac{c}{n} - \frac{1}{nd}, \frac{c}{n} + \frac{1}{(n-d)n}\right) \setminus \{\pi\frac{c}{n}\}$$

with integers c, d satisfying $1 \leq c, d \leq (n-1)$, $\gcd(n, c) = 1$ and $cd = 1 \mod n$. In these cases the reduced phase space $P(\mu_0(x))$ is symplectomorphic to $\mathbb{CP}^{n-1}$ endowed with a multiple of the Fubini-Study symplectic structure.

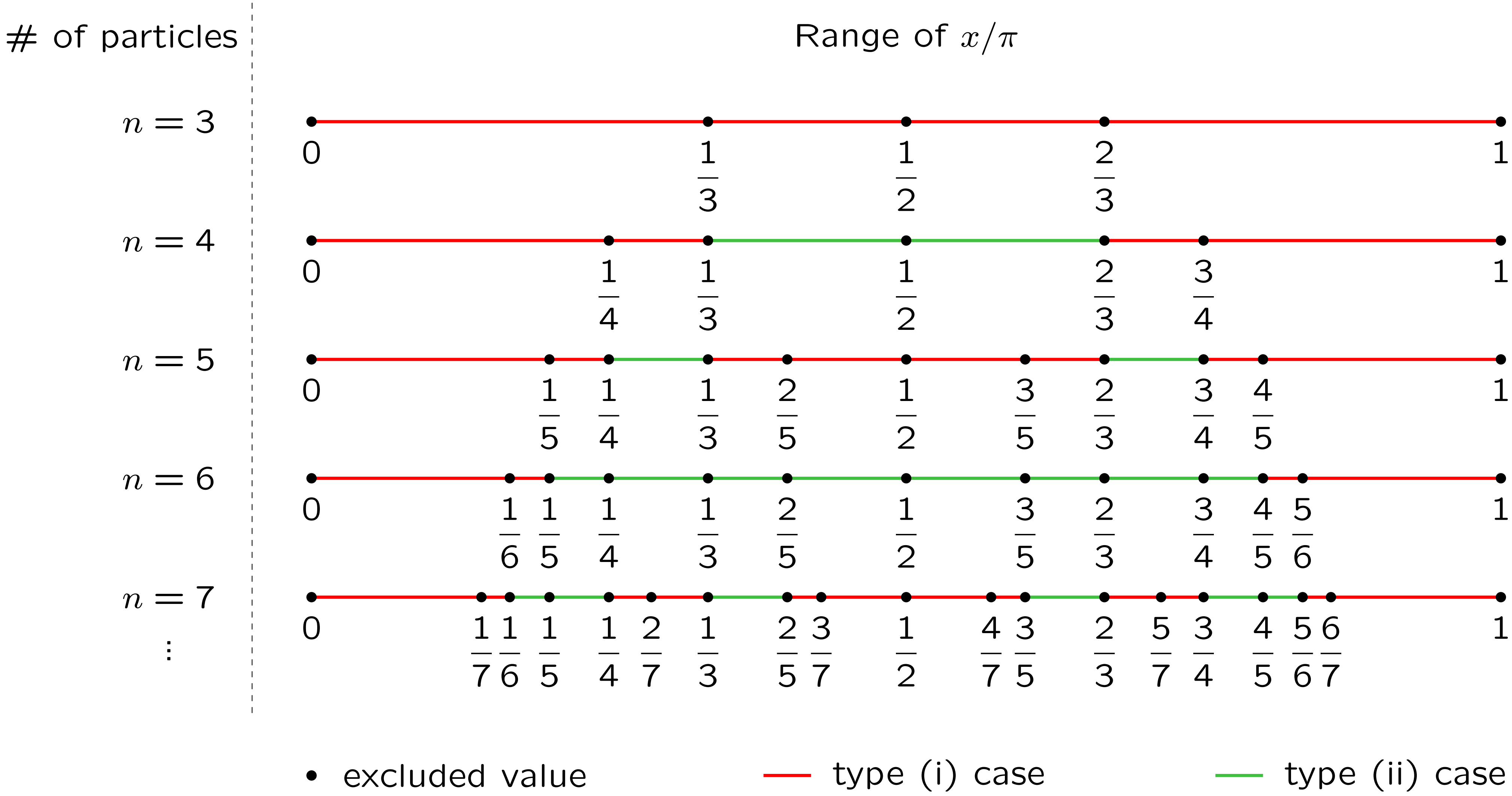
- The result was obtained by determining the possible spectra of the matrix B satisfying $ABA^{-1}B^{-1} = \mu_0(x)$.

- In the type (i) cases we found that the "Delzant polytope" is a simplex.

- The existence of type (ii) cases was not anticipated.

- In the previously studied type (i) case [Ruijsenaars 95, van Diejen-Vinet 98, Gorsky-Nekrasov 95, Feher-Klimcik 2012] $c = 1$ and x was restricted to $(0, \pi/n)$.

Illustration of type (i) and type (ii) cases



“Free” Hamiltonians on the double and their reductions

For any $\mathcal{H} \in C^\infty(G)^G$, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the invariant functions on D given by $\mathcal{H}_1(A,B) := \mathcal{H}(A)$ and $\mathcal{H}_2(A,B) := \mathcal{H}(B)$. Then $\{\mathcal{H}_1\}$ and $\{\mathcal{H}_2\}$ form two Abelian Poisson algebras on D. One can easily write down the corresponding quasi-Hamiltonian flows on D.

By reduction, one obtains two Abelian Poisson algebras on each reduced phase space $P(\mu_0)$:

$$\mathcal{C}^a := \{\hat{\mathcal{H}}_1 \,|\, \mathcal{H} \in C^\infty(G)^G\}, \quad \mathcal{C}^b := \{\hat{\mathcal{H}}_2 \,|\, \mathcal{H} \in C^\infty(G)^G\}.$$

Abelian algebras are interchanged under ‘duality symplectomorphism’ (of order 4) S_P that descends from automorphism S_D of the double, $S_D : (A,B) \mapsto (B^{-1}, BAB^{-1})$.

Consequence: The ‘configuration space’ $\mathcal{A}_x$ described later (page 12) is **THE SAME** as the range of the action variables.

Basic "spectral functions" on $SU(n)$

Simplex: $\Delta := \left\{ (\xi_1, ..., \xi_{n-1}) \in \mathbb{R}^{n-1} \,\middle|\, \xi_j \geq 0,\ j = 1, ..., n-1,\ \sum_{j=1}^{n-1} \xi_j \leq \pi \right\}$

$n \times n$ matrices: $\Lambda_k := \sum_{j=1}^{k} E_{j,j} - \frac{k}{n} \mathbf{1}_n, \quad k = 1, ..., n-1$

Any element of $G = SU(n)$ is conjugate to $\delta(\xi) := \exp\left(-2\mathrm{i} \sum_{k=1}^{n-1} \xi_k \Lambda_k\right)$ for unique $\xi \in \Delta$. Hence, we can define conjugation invariant functions Ξ_k on G by setting

$$\Xi_k(\delta(\xi)) := \xi_k, \quad \forall \xi \in \Delta, \quad k = 1, ..., n-1.$$

"**Spectral functions**" Ξ_k are only continuous at G_{sing}, but restrictions to G_{reg} belong to $C^\infty(G_{\text{reg}})^G$. G_{reg} is mapped onto **interior** of Δ by $(\Xi_1, \ldots, \Xi_{n-1})$.

Crucial fact: invariant functions $\alpha_k(A, B) := \Xi_k(A)$ and $\beta_k(A, B) := \Xi_k(B)$ **generate** 2π**-periodic flows on the regular part of the double** $D = G \times G$.

Reduction applied to $(\alpha_1, \ldots, \alpha_{n-1}) : D \to \Delta$ and $(\beta_1, \ldots, \beta_{n-1}) : D \to \Delta$ yields **toric moment maps** on the reduced phase space $P(\mu_0)$ if $P(\mu_0)$ is smooth, has dimension $2(n-1)$ and $\mu^{-1}(\mu_0) \subset G_{\text{reg}} \times G_{\text{reg}}$.

We found all cases when the above conditions hold.

The "configuration space"

It is convenient to map $\mathbb{R}^{n-1}$ onto the hyperplane

$E := \{\xi \in \mathbb{R}^n \mid \xi_1 + \cdots + \xi_n = \pi\}$ that contains $\Delta = \{\xi \in E \mid \xi_\ell \geq 0,\ \forall \ell = 1, \ldots, n\}$.

Let $\delta(\xi) = gBg^{-1}$ and $g\mu_0(x)g^{-1} = e^{2\mathrm{i}x}\mathbf{1}_n + (e^{2\mathrm{i}(1-n)x} - e^{2\mathrm{i}x})vv^\dagger$. **For regular ξ, the constraint $ABA^{-1}B^{-1} = \mu_0(x)$ implies**

$$|v_\ell|^2 = \frac{\sin(x)}{\sin(nx)} \prod_{\substack{j=1\\ j\neq\ell}}^{n} \frac{e^{-\mathrm{i}x}\delta_j - e^{\mathrm{i}x}\delta_\ell}{\delta_j - \delta_\ell} = \frac{\sin(x)}{\sin(nx)} \prod_{j=\ell+1}^{\ell+n-1} \frac{\sin(\sum_{k=\ell}^{j-1}\xi_k - x)}{\sin(\sum_{k=\ell}^{j-1}\xi_k)} := z_\ell(\xi, x)$$

and the task is to find the "configuration space" $\mathcal{A}_x$, provided by the closure of
$\mathcal{A}_x^{\mathrm{reg}} = \{\xi \in \Delta^{\mathrm{reg}} \mid z_\ell(\xi, x) \geq 0,\ \ell = 1, \ldots, n\}$.

Using periodic convention $\xi_j = \xi_{j+n}$ $(\forall j \in \mathbb{Z})$, we find that for

$$c\frac{\pi}{n} < x < (c+1)\frac{\pi}{n}, \qquad (c = 0, \ldots, n-1)$$

$$\mathcal{A}_x = \{\xi \in E \mid \xi_\ell + \cdots + \xi_{\ell+c-1} \leq x,\ \forall \ell = 1, \ldots, n\} \cap \{\xi \in E \mid \xi_\ell + \cdots + \xi_{\ell+c} \geq x,\ \forall \ell = 1, \ldots n\}$$

Thus $\mathcal{A}_x$ is intersection of two polyhedra. It is contained in the closed simplex Δ.
(If $c = 1$ or $c = n-1$ then one polyhedron occurs, and it lies inside Δ.)

In type (i) cases $\mathcal{A}_x$ does not reach the boundary of Δ. This happens when x is near enough to $\pi\frac{c}{n}$ for $\gcd(c, n) = 1$. In these cases one of the two polyhedra is a simplex, which is contained in the other polyhedron and inside Δ.

Preparation for local description of the reduced system

Pick any x for which $e^{2\mathrm{i}xm} \neq 1$ for all $m = 1, 2, \ldots, n$. Consider domain $\mathcal{A}_x^+$ containing those regular ξ for which $z_\ell(\xi, x) > 0$ for all $\ell = 1, \ldots, n$.

Then take $v_\ell(\xi, x) := \sqrt{z_\ell(\xi, x)}$, and using $v \equiv v(\xi, x)$ introduce the matrix $g := g_x(\xi)$ having the elements

$$g_{nn} := v_n, \;\; g_{jn} := -g_{nj} := v_j, \;\; g_{jl} := \delta_{jl} - \frac{v_j v_l}{1 + v_n}, \;\; \forall j, l = 1, \ldots, n-1.$$

Finally, with $(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_{n-1}}) \in \mathbb{T}^{n-1}$ prepare

$$\varrho := \mathrm{diag}(e^{-\mathrm{i}\theta_1}, e^{\mathrm{i}(\theta_1 - \theta_2)}, e^{\mathrm{i}(\theta_2 - \theta_3)}, \ldots, e^{\mathrm{i}(\theta_{n-2} - \theta_{n-1})}, e^{\mathrm{i}\theta_{n-1}}).$$

Then we have the unitary 'local RS Lax matrix'

$$\mathcal{L}_x^{\mathrm{loc}}(\xi, \theta)_{j\ell} = \frac{\sin(nx)}{\sin(x)} \frac{e^{\mathrm{i}x} - e^{-\mathrm{i}x}}{e^{\mathrm{i}x}\delta_j(\xi)\delta_\ell(\xi)^{-1} - e^{-\mathrm{i}x}} v_j(\xi, x) v_\ell(\xi, -x) \varrho(\theta)_\ell.$$

Note that x matters only modulo π and $\mathcal{A}_x^+ = \mathcal{A}_{\pi - x}^+ \equiv \mathcal{A}_{-x}^+$.

Local Theorem. For any generic x, the set

$$\left\{ \left(g_x(\xi)^{-1} \mathcal{L}_x^{\mathrm{loc}}(\xi,\theta) g_x(\xi), g_x(\xi)^{-1}\delta(\xi) g_x(\xi) \right) \,\middle|\, (\xi, e^{\mathrm{i}\theta}) \in \mathcal{A}_x^+ \times \mathbb{T}^{n-1} \right\} \subset G\times G$$

defines a cross-section of the orbits of $G_{\mu_0(x)}$ in the open submanifold $\beta^{-1}(\mathcal{A}_x^+) \cap \mu^{-1}(\mu_0(x))$ of the constraint surface. The parametrization by $(\xi, e^{\mathrm{i}\theta}) \in \mathcal{A}_x^+ \times \mathbb{T}^{n-1}$ induces Darboux coordinates on corresponding submanifold of reduced phase space: we have $\omega^{\mathrm{loc}} = \sum_{k=1}^{n-1} \mathrm{d}\theta_k \wedge \mathrm{d}\xi_k$.

On this submanifold, which is **dense** in the full reduced phase space, the Poisson commuting reduced Hamiltonians descending from the class functions of A in $(A,B) \in G\times G$ become the class functions of $\mathcal{L}_x^{\mathrm{loc}}(\xi,\theta)$. The reduction of the function $\Re(\mathrm{tr}\,(A))$ provides the RS Hamiltonian

$$H_x^{\mathrm{loc}}(\xi,\theta) = \sum_{j=1}^{n} \cos(\theta_j - \theta_{j-1}) \sqrt{\prod_{k=j+1}^{j+n-1} \left[1 - \frac{\sin^2 x}{\sin^2(\sum_{m=j}^{k-1} \xi_m)} \right]}$$

The α and β images of reduced phase space give closure of $\mathcal{A}_x^+ \subset \Delta$.

(Here we employed the conventions $\theta_0 = \theta_n = 0$, $\xi_n = \pi - \xi_1 - \cdots - \xi_{n-1}$ and $\xi_{k+n} = \xi_k$.)

For interpretation, put $\delta_k = e^{2\mathrm{i}q_k}$, $\varrho_k = e^{-\mathrm{i}p_k}$, $q_{k+1} - q_k = \xi_k$, $\Big(\prod_{k=1}^{n} \delta_k = \prod_{k=1}^{n} \varrho_k = 1\Big)$.

Then $H_x^{\mathrm{loc}}(q,p) = \sum_{j=1}^{n} \cos(p_j) \sqrt{\prod_{k\neq j}\Big[1 - \frac{\sin^2 x}{\sin^2(q_j - q_k)}\Big]}$ and, after a conjugation, the local Lax matrix becomes

$$L_x^{\mathrm{loc}}(q,p)_{j,\ell} = \frac{\sin(nx)}{\sin(x)} \frac{\sin(x)}{\sin(q_j - q_\ell + x)} v_j(\xi, x) v_\ell(\xi, -x) \varrho_\ell$$

$$v_j(\xi, x) = \Big[\frac{\sin(x)}{\sin(nx)} \prod_{\substack{k=1\\k\neq j}}^{n} \frac{\sin(q_k - q_j - x)}{\sin(q_k - q_j)}\Big]^{\frac{1}{2}} = \Big[\frac{\sin(x)}{\sin(nx)} \prod_{k=j+1}^{j+n-1} \frac{\sin(\sum_{m=j}^{k-1} \xi_m - x)}{\sin(\sum_{m=j}^{k-1} \xi_m)}\Big]^{\frac{1}{2}}$$

In the type (i) cases, fix integers $1 \leq c, d \leq (n-1)$ s.t. $\mathrm{gcd}(c,n) = 1$ and $cd = 1$ modulo n. Then the parameter x can vary as

$$\Big(\frac{c}{n} - \frac{1}{nd}\Big)\pi < x < \frac{c\pi}{n} \qquad \text{or} \qquad \frac{c\pi}{n} < x < \Big(\frac{c}{n} + \frac{1}{(n-d)n}\Big)\pi.$$

In the above two cases $M := c\pi - nx > 0$ or $M < 0$, respectively, and $\xi \in \mathcal{A}_x$ satisfies

$$\mathrm{sgn}(M)(\xi_j + \cdots + \xi_{j+c-1} - x) \geq 0, \quad j = 1, \ldots, n.$$

Thus, for $M > 0$ and $M < 0$, the '**distances of the c-th neighbours**' are subject to

$$q_{j+c} - q_j \geq x \quad \text{and respectively to} \quad q_{j+c} - q_j \leq x, \qquad \forall j.$$

Here, $q_{k+n} = q_k + \pi$. The simplex $\mathcal{A}_x$ lies in the interior of Δ.

Turning to the second part, we now embed the local phase space into $\mathbb{CP}^{n-1}$.

For this, we introduce the mapping $\mathcal{E}\colon \mathcal{A}_x^+ \times \mathbb{T}^{n-1} \to \mathbb{C}^n, \quad (\xi, e^{\mathrm{i}\theta}) \mapsto (u_1, \ldots, u_n)$ with the complex coordinates having the squared absolute values

$$|u_j|^2 = \mathrm{sgn}(M)(\xi_j + \cdots + \xi_{j+c-1} - x), \quad j = 1, \ldots, n,$$

and arguments $\arg(u_j) = \mathrm{sgn}(M) \sum_{k=1}^{n-1} W_{j,k}\theta_k$, $j = 1, \ldots, n-1$, $\arg(u_n) = 0$. We have

$$|u_j|^2 = \begin{cases} \mathrm{sgn}(M)\big(\sum_{k=1}^{n-1} T_{j,k}\xi_k - x\big), & \text{if } 1 \leq j \leq n-p, \\ \mathrm{sgn}(M)\big(\sum_{k=1}^{n-1} T_{j,k}\xi_k - x + \pi\big), & \text{if } n-p < j \leq n-1 \end{cases}$$

with an integer matrix $T \in \mathrm{SL}(n-1, \mathbb{Z})$, and take W to be inverse-transpose of T. (We determined T and T^{-1} explicitly.) The image of $\mathcal{E}$ lies in

$$S^{2n-1}_{|M|} = \{(u_1, \ldots, u_n) \in \mathbb{C}^n \mid |u_1|^2 + \cdots + |u_n|^2 = |M|\},$$

which engenders $\mathbb{CP}^{n-1} = S^{2n-1}_{|M|}/\mathrm{U}(1)$, with projection $\pi_{|M|}\colon S^{2n-1}_{|M|} \to \mathbb{CP}^{n-1}$.

As $\mathcal{E}^*\Big(\mathrm{i} \sum_{j=1}^{n} d\bar{u}_j \wedge du_j\Big) = \sum_{k=1}^{n-1} d\theta_k \wedge d\xi_k$ holds, we obtained **symplectic embedding**

$$\pi_{|M|} \circ \mathcal{E} : \mathcal{A}_x^+ \times \mathbb{T}^{n-1} \to \mathbb{CP}^{n-1}$$

with respect to $\omega^{\mathrm{loc}} = \sum_{j=1}^{n-1} d\theta \wedge d\xi_j$ and the re-scaled Fubini-Study form $|M|\omega_{\mathrm{FS}}$. The image is the dense, open submanifold where no homogeneous coordinate can vanish.

Result about direct construction of trigonometric RS systems on $\mathbb{CP}^{n-1}$

Theorem. Define the diagonal unitary matrix $D = \mathrm{diag}(D_1, \ldots, D_{n-1}, 1)$ with

$$D_j = \exp\left(\mathrm{i} \sum_{k=1}^{n-1} W_{j,k}\theta_k\right), \quad j = 1, \ldots, n-1.$$

Then, in every type (i) case, there exists a smooth function $L^x : \mathbb{CP}^{n-1} \to \mathrm{SU}(n)$ that satisfies the following relation:

$$(L^x \circ \pi_{|M|} \circ \mathcal{E})(\xi, \theta) = D(\theta)^{-1} L_x^{\mathrm{loc}}(\xi, \theta) D(\theta), \quad \forall (\xi, e^{\mathrm{i}\theta}) \in \mathcal{A}_x^+ \times \mathbb{T}^{n-1},$$

which means that L^x is an extension of the local Lax matrix $D^{-1} L_x^{\mathrm{loc}} D$ to $\mathbb{CP}^{n-1}$.

Corollary. The symmetric functions of the *global Lax matrix* L^x define an integrable system on $\mathbb{CP}^{n-1}$, whose main Hamiltonian extends the local RS Hamiltonian H_x^{loc}.

We have this extension in explicit form as well. Next I sketch the crux of the proof, and then give the analogous result in the elliptic case.

The direct construction was inspired by Ruijsenaars' work [RIMS 95], which dealt with the case $0 < x < \pi/n$, and his remarks on the corresponding elliptic case.

To explain the crux, first note that $C^\infty(\mathbb{CP}^{n-1}) = C^\infty(S^{2n-1}_{|M|})^{\mathrm{U}(1)}$. Thus the squared absolute values $|u_j|^2$ give rise to smooth functions on $\mathbb{CP}^{n-1}$, and the same is true for the components ξ_k, which be written as affine combinations of the $|u_j|^2$.

Consider 'building block' $v_j(\xi, x)$ of local Lax matrix. We have $v_j(\xi, x) = |u_j| w_j(\xi, x)$, where $w_j(\xi, x)$ is the function

$$w_j(\xi, x) = \left[\frac{\sin(|u_j|^2)}{|u_j|^2} \frac{\mathrm{sgn}(M)\sin(x)}{\sin(nx)\sin(\sum_{k=j}^{j+p-1}\xi_k)} \prod_{\substack{m=j+1\\(m\neq j+p)}}^{j+n-1} \frac{\sin(\sum_{k=j}^{m-1}\xi_k - x)}{\sin(\sum_{k=j}^{m-1}\xi_k)} \right]^{\frac{1}{2}}.$$

The point to notice is that w_j extends to a smooth function on $\mathbb{CP}^{n-1}$. Inspecting all building blocks, we find that the local Lax matrix exhibits the following structure:

$$L_x^{\mathrm{loc}}(\xi, 0)_{j,\ell} = \begin{cases} \Lambda^x_{j,j+p}(\xi), & \text{if } 1 \leq j \leq n-p,\ \ell = j+p \\ \Lambda^x_{j,j-(n-p)}(\xi), & \text{if } n-p < j \leq n,\ \ell = j-(n-p), \\ |u_j||u_{\ell-p+n}|\Lambda^x_{j,\ell}(\xi), & \text{if } 1 \leq j \leq n,\ 1 \leq \ell \leq p,\ \ell \neq j-(n-p), \\ |u_j||u_{\ell-p}|\Lambda^x_{j,\ell}(\xi), & \text{if } p < \ell \leq n,\ \ell \neq j+p. \end{cases}$$

where the $\Lambda^x_{j,\ell}(\xi)$ extend to smooth functions on $\mathbb{CP}^{n-1}$. The absolute values are not smooth functions (at the origin), but they appear quadratically. Everything will be fine if we can "engineer" replacements like $|u_j||u_{\ell-p}| \longrightarrow \bar{u}_j u_{\ell-p}$ since on the r.h.s we have a U(1) invariant smooth function on $S^{2n-1}_{|M|}$. This is precisely what is achieved by conjugating $L_x^{\mathrm{loc}}(\xi, \theta) = L_x^{\mathrm{loc}}(\xi, 0)\varrho(\theta)$ by the phase matrix $D(\theta)$.

Elliptic preparations

Let ω, ω' stand for the half-periods of the Weierstrass elliptic function $\wp$,

$$\wp(z;\omega,\omega') = z^{-2} + \sum_{\substack{m,m'=-\infty \\ (m,m')\neq(0,0)}}^{\infty} \left((z-\Omega_{m,m'})^{-2} - \Omega_{m,m'}^{-2}\right),$$

with $\Omega_{m,m'} = 2m\omega + 2m'\omega'$. We choose $\omega, -\mathrm{i}\omega' \in (0,\infty)$, which ensures that $\wp$ is positive on the real axis. Next, introduce the "s-function" by the formula

$$\mathrm{s}(z;\omega,\omega') = a^{-1}\sin(az)\prod_{m=1}^{\infty}\left(1 - \frac{\sin^2(az)}{\sin^2(2am\omega')}\right)$$

with $a = \pi/(2\omega)$. An important identity connecting $\wp$ and s is

$$\frac{\mathrm{s}(z+y)\,\mathrm{s}(z-y)}{\mathrm{s}^2(z)\,\mathrm{s}^2(y)} = \wp(y) - \wp(z).$$

The s-function is odd, satisfies $\mathrm{s}(\pi/a - z) = \mathrm{s}(z)$, has simple zeros at $\Omega_{m,m'}$, $m, m' \in \mathbb{Z}$ and enjoys the scaling property

$$\mathrm{s}(tz;t\omega,t\omega') = t\,\mathrm{s}(z;\omega,\omega'),$$

which permits to work with $a = 1$ ($\omega = \pi/2$). In the trigonometric limit,

$$\lim_{-\mathrm{i}\omega'\to\infty}\wp(z;\pi/2,\omega') = \frac{1}{\sin^2(z)} - \frac{1}{3}, \qquad \lim_{-\mathrm{i}\omega'\to\infty}\mathrm{s}(z;\pi/2,\omega') = \sin(z).$$

Type (i) compact forms of the elliptic RS system

Since $\mathsf{s}(z)$ and $\sin(z)$ have matching properties, the following variant Ruijsenaars' [1986] elliptic Lax matrix is well-behaved on the type (i) local phase space $\mathcal{A}_x^+ \times \mathbb{T}^{n-1}$:

$$L_x^{\mathrm{loc}}(\xi,\theta|\lambda)_{j,\ell} = \frac{\mathsf{s}(nx)}{\mathsf{s}(x)}\frac{\mathsf{s}(x)}{\mathsf{s}(\lambda)}\frac{\mathsf{s}(q_j - q_\ell + \lambda)}{\mathsf{s}(q_j - q_\ell + x)} v_j(\xi,x)v_\ell(\xi,-x)\varrho(\theta)_\ell,$$

where $\lambda \in \mathbb{C} \setminus \{\Omega_{m,m'} : m,m' \in \mathbb{Z}\}$ is a spectral parameter, $v_\ell(\xi,\pm x) = \sqrt{z_\ell(\xi,\pm x)}$ with

$$z_\ell(\xi,x) = \frac{\mathsf{s}(x)}{\mathsf{s}(nx)} \prod_{m=\ell+1}^{\ell+n-1} \frac{\mathsf{s}(\sum_{k=\ell}^{m-1}\xi_k - x)}{\mathsf{s}(\sum_{k=\ell}^{m-1}\xi_k)} = \frac{\mathsf{s}(x)}{\mathsf{s}(nx)} \prod_{\substack{m=1\\ m\neq\ell}}^{n} \frac{\mathsf{s}(q_m - q_\ell - x)}{\mathsf{s}(q_m - q_\ell)}$$

Theorem. There exists a smooth, spectral parameter dependent elliptic Lax matrix $L^x(\cdot|\lambda)$ on $\mathbb{CP}^{n-1}$ which is an extension of $L_x^{\mathrm{loc}}(\xi,\theta|\lambda)$ since it satisfies

$$L^x(\pi_{|M|} \circ \mathcal{E}(\xi,\theta)|\lambda) = D(\theta)^{-1} L_x^{\mathrm{loc}}(\xi,\theta|\lambda) D(\theta), \quad \forall (\xi, e^{\mathrm{i}\theta}) \in \mathcal{A}_y^+ \times \mathbb{T}^{n-1},$$

where D and $\mathcal{E} \circ \pi_{|M|} : \mathcal{A}_x^+ \times \mathbb{T}^{n-1} \to \mathbb{CP}^{n-1}$ are the same as in the trigonometric case.

We have $\mathrm{sgn}(\mathsf{s}(nx))\Re(\mathrm{tr}\, L_x^{\mathrm{loc}}(\xi,\theta)) = H_x^{\mathrm{loc}}(\xi,\theta)$ with the elliptic RS ($\mathrm{IV_b}$) Hamiltonian:

$$H_x^{\mathrm{loc}}(\xi,\theta) = \sum_{j=1}^{n} \cos(\theta_j - \theta_{j-1}) \sqrt{\prod_{m=j+1}^{j+n-1} \Big(\mathsf{s}(x)^2(\wp(x) - \wp(\textstyle\sum_{k=\ell}^{m-1}\xi_k))\Big)}.$$

Conclusion

My research is focused on applications of Hamiltonian reduction. This links integrable systems to a host of interesting subjects. I explored several many-body systems and their duality relations in this framework.

Projects for the near future:

- New cases of duality associated with BC_n (I. Marshall)
- Quantization of new compact RS systems (T.F. Görbe)

Main open problems:

- How to deal with (duality for) 'relativistic Toda'?
- How to obtain the hyperbolic RS system by reduction?

REFS: L.F. and T.J. Kluck: New compact forms of the trigonometric RS system, Nucl. Phys. B882 (2014) 97-127

L.F. and T.F. Görbe: Trigonometric and elliptic RS systems on the complex projective space, arXiv:1605.09736