Calogero-Sutherland type models from Hamiltonian reduction

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Talk mainly based on: `preprint arXiv:0705.1998`

Our purpose is to systematically develop the Hamiltonian reduction approach to C-S type integrable models both classically and quantum mechanically. This is essentially a chapter in harmonic analyis, but in that field the classical mechanical aspects are not considered.

Among others, our work builds on and tries to further develop the results in the landmark contributions of Olshanetsky-Perelomov (1976,1978), Kazhdan-Kostant-Sternberg (1978), Etingof-Frenkel-Kirillov (95), using standard harmonic analysis, e.g. Helgason (72).

References on our project

• Spin Calogero models obtained from dynamical r-matrices and geodesic motion, *Nucl. Phys.* **B734**, *304-325 (2006)*

• Spin Calogero models and dynamical r-matrices, *Bulg. J. Phys.* **33**, *261-272 (2006)*

• Spin Calogero models associated with Riemannian symmetric spaces of negative curvature, *Nucl. Phys.* **B751**, *436-458 (2006)*

• A class of Calogero type reductions of free motion on a simple Lie group, *Lett. Math. Phys.* **79**, *263-277 (2007)*

• Hamiltonian reductions of free particles under polar actions of compact Lie groups, *arXiv:0705.1998*, to appear in *Theor. Math. Phys.*

• On the self-adjointness of certain reduced Laplace-Beltrami operators, *arXiv:0707.2708*

• There are further papers in preparation devoted to computing spectra and dealing with other aspects and examples.

PLAN OF THE PRESENTATION

- **Reminder on polar actions of Lie groups**
- **Classical Hamiltonian reduction**
- **Quantum Hamiltonian reduction**
- **The self-adjointness of the reduced Hamiltonians**
- **Twisted quantum spin Sutherland models**
- **More examples of spin Calogero-Sutherland type models**
- **Spinless BC_n model with 3 independent coupling constants**
- **Concluding remarks**

POLAR GROUP ACTIONS

Consider a smooth, isometric action of a compact Lie group, G, on a **connected, complete** Riemannian manifold, Y with metric η. The action is called polar if it admits a **connected**, **closed**, **imbedded** (regular) **submanifold** $\Sigma \subset Y$ that intersects **all** G-orbits **orthogonally**. Such a submanifold Σ is a 'section' for the action.

For polar actions, there is a unique section through any point $y \in Y$ with **principal isotropy** type, given by $\exp\Big((T_y(G.y))^{\perp}\Big)$. The action is called hyperpolar if the sections are **flat** in the induced metric.

Following earlier works by L. Conlon (1971) and J. Szenthe (1984) on hyperpolar and polar actions, motivated by pioneering works of R. Bott and H. Samelson (1958) and R. Hermann (1960), polar actions were defined and investigated systematically by **R. Palais and C.-L. Terng**, *Trans. Amer. Math. Soc. 300, 771-789 (1987)*.

Since the Palais-Terng paper, (hyper)polar actions (especially on symmetric spaces) have been much studied in differential geometry.

SOME EXAMPLES OF HYPERPOLAR ACTIONS

• 1. The standard action of $SO(n)$ on the Euclidean space $\mathbb{R}^n$ is hyperpolar. The sections are the straight lines through the origin.

• 2. The adjoint action of a connected compact simple Lie group G on itself is hyperpolar. The sections are just the maximal tori. The adjoint representation of G on the Lie algebra T_eG is also hyperpolar, with the sections being the Cartan subalgebras.

• 3. Let X be a non-compact simple Lie group with finite centre and maximal compact subgroup G. The induced actions of G on the symmetric spaces X/G and on $T_{[e]}(X/G)$ are hyperpolar.

• 4. Let Y be a compact, connected, semisimple Lie group carrying the Riemannian metric induced by a multiple of the Killing form. Take G to be any **symmetric subgroup** of $Y \times Y$, fixed by some involution σ. The action of G on Y, defined by $\phi_{(a,b)} \in \text{Diff}(Y)$ as

$$\phi_{(a,b)}(y) := ayb^{-1}, \quad \forall y \in Y, \quad (a,b) \in G \subset Y \times Y$$

is hyperpolar. The sections are provided by certain tori, $A \subset Y$.

GENERALIZED POLAR COORDINATES

$\check{Y} \subset Y$: open, dense submanifold of 'regular elements' of principal isotropy type w.r.t. polar action $G \ni g \mapsto \phi_g \in \mathrm{Diff}(Y)$

$\check{\Sigma}$: a connected component of $\hat{\Sigma} := \check{Y} \cap \Sigma$ for fixed section Σ

K: isotropy group of the elements of $\hat{\Sigma}$

One has diffeomorphism $\check{Y} \simeq \check{\Sigma} \times G/K$, whereby $\check{Y} \ni y \simeq \phi_{gK}(q)$ with $q \in \check{\Sigma}$ and $gK \in G/K$. $\check{\Sigma}$ and G/K are radial and orbital parts.

Induced metric η_{red} on smooth part of reduced configuration space $\check{Y}_{\mathrm{red}} := \check{Y}/G$ is equivalent to metric $\eta_{\check{\Sigma}}$ on submanifold $\check{\Sigma} \subset \check{Y}$

For $q \in \check{\Sigma}$, one has orthogonal decomposition $T_q\check{Y} = T_q\check{\Sigma} \oplus T_q(G.q)$. Choosing an invariant scalar product $\mathcal{B}$ on $\mathcal{G}$, $\mathcal{G} = \mathcal{K} \oplus \mathcal{K}^\perp$ where $\mathcal{G} = \mathrm{Lie}(G)$, $\mathcal{K} := \mathrm{Lie}(K)$. Then $\mathcal{K}^\perp$ is a model of $T_q(G.q)$ by $\mathcal{K}^\perp \ni \xi \mapsto \xi_Y(q)$ with vector field ξ_Y on Y.

The induced metric $\eta_{G.q}$ on the submanifold $G.q \subset Y$ is encoded by the (K-equivariant, symmetric, positive definite) inertia operator $\mathcal{I}(q) \in \mathrm{End}(\mathcal{K}^\perp)$ as $\eta_q(\xi_Y(q), \zeta_Y(q)) = \mathcal{B}(\mathcal{I}(q)\xi, \zeta) \quad \forall \xi, \zeta \in \mathcal{K}^\perp$

Data $\eta_{\mathrm{red}} \simeq \eta_{\check{\Sigma}}$ and $\mathcal{I}$ determine the Riemannian metric η on Y.

In 'radial-angular' coordinates $\check{\Sigma} \times G/K$, metric η is block-diagonal.

Classical Hamiltonian reduction - definitions

We fix a coadjoint orbit $(\mathcal{O}, \omega)$ of G, and start from the extended Hamiltonian system $(\check{\mathcal{P}}^{\text{ext}}, \Omega^{\text{ext}}, \mathcal{H}^{\text{ext}})$ of the free motion on $(\check{Y}, \eta)$:

$$\check{\mathcal{P}}^{\text{ext}} := T^*\check{Y} \times \mathcal{O} = \{(\alpha_y, \xi) \,|\, \alpha_y \in T_y^*\check{Y},\, y \in \check{Y},\, \xi \in \mathcal{O}\}$$

$$\Omega^{\text{ext}}(\alpha_y, \xi) = (d\theta_{\check{Y}})(\alpha_y) + \omega(\xi), \quad \mathcal{H}^{\text{ext}}(\alpha_y, \xi) := \frac{1}{2}\eta_y^*(\alpha_y, \alpha_y)$$

with the canonical 1-form $\theta_{\check{Y}}$ of $T^*\check{Y}$ and the metric η_y^* on $T_y^*\check{Y}$.
Action of G on $\check{\mathcal{P}}^{\text{ext}}$ is generated by momentum map $\Psi : \check{\mathcal{P}}^{\text{ext}} \to \mathcal{G}^*$
$\Psi(\alpha_y, \xi) = \psi(\alpha_y) + \xi$ with $\psi : T^*\check{Y} \to \mathcal{G}^*$ generating action on $T^*\check{Y}$.

Interested in reduced Hamiltonian system at the value $\Psi = 0$:

$$(\check{\mathcal{P}}_{\text{red}}, \Omega_{\text{red}}, \mathcal{H}_{\text{red}}) \quad \text{where} \quad \check{\mathcal{P}}_{\text{red}} = \check{\mathcal{P}}^{\text{ext}}//_0 G := \check{\mathcal{P}}^{\text{ext}}_{\Psi=0}/G$$

This is the same as (singular) Marsden-Weinstein reduction of $T^*\check{Y}$ at $\mu \in -\mathcal{O}$.

Result contains (singular) reduced orbit $\mathcal{O}_{\text{red}} = \mathcal{O}//_0 K \simeq (\mathcal{O} \cap \mathcal{K}^\perp)/K$ equipped with reduced symplectic form ω_{red}. Here $K \subset G$ acts naturally with momentum map $\mathcal{O} \ni \xi \mapsto \xi|\mathcal{K}$ and we identify $\mathcal{G} \simeq \mathcal{G}^*$ and $\mathcal{G}^* \supset \mathcal{K}^0 \simeq \mathcal{K}^\perp \subset \mathcal{G}$ by means of invariant scalar product $\mathcal{B}$ on $\mathcal{G}$.

Result of the classical Hamiltonian reduction

The reduced configuration space $\check{Y}_{\text{red}}$ inherites the Riemannian metric η_{red}. Let η^*_{red} denote the metric and $\theta_{\check{Y}_{\text{red}}}$ the natural 1-form on $T^*\check{Y}_{\text{red}}$. The next theorem follows from general results of S. Hochgerner: math.SG/0411068 on reduced cotangent bundles. With B.G. Pusztai, we gave a direct proof in arXiv:0705.1998.

Theorem 1. *Consider a polar G-action on (Y, η) and fix a connected component $\check{\Sigma}$ of the regular elements of a section Σ. Then the reduced system $(\check{P}_{\text{red}}, \Omega_{\text{red}}, \mathcal{H}_{\text{red}})$ can be identified as*

$$\check{P}_{\text{red}} = T^*\check{Y}_{\text{red}} \times \mathcal{O}_{\text{red}} = \{(p_q, [\xi]) \,|\, p_q \in T^*_q\check{Y}_{\text{red}},\ q \in \check{Y}_{\text{red}},\ [\xi] \in \mathcal{O}_{\text{red}}\}$$

equipped with the product (stratified) symplectic structure

$$\Omega_{\text{red}}(p_q, [\xi]) = (d\theta_{\check{Y}_{\text{red}}})(p_q) + \omega_{\text{red}}([\xi])$$

and the reduced Hamiltonian induced by the free kinetic energy

$$\mathcal{H}_{\text{red}}(p_q, [\xi]) = \frac{1}{2}\eta^*_{\text{red}}(p_q, p_q) + \frac{1}{2}\mathcal{B}(\mathcal{I}_q^{-1}\xi, \xi)$$

where $[\xi] = K.\xi \subset \mathcal{O} \cap \mathcal{K}^\perp$ and $\mathcal{I}_q \in GL(\mathcal{K}^\perp)$ is the K-equivariant inertia operator for $q \in \check{\Sigma} \simeq \check{Y}_{\text{red}}$.

Remark: This gives a natural Hamiltonian system if $\mathcal{O}_{\text{red}}$ is a 1-point space.

Definition of quantum Hamiltonian reduction

Quantized analogue of $P^{\text{ext}} = T^*Y \times \mathcal{O}$ is $L^2(Y, V, d\mu_Y)$, where we replace the orbit $\mathcal{O}$ by unitary representation $\rho : G \to U(V)$ on finite dimensional complex Hilbert space V with scalar product $(\ ,\ )_V$. The scalar product of V-valued wave functions reads

$$(\mathcal{F}_1, \mathcal{F}_2) = \int_Y (\mathcal{F}_1, \mathcal{F}_2)_V d\mu_Y$$

where $d\mu_Y$ is the measure induced by Riemannian metric η on Y.

Denote by Δ_Y^0 the Laplace-Beltrami operator Δ_Y of (Y, η) *on the domain* $C_c^\infty(Y, V) \subset L^2(Y, V, d\mu_Y)$ containing the smooth V-valued functions of compact support. Δ_Y^0 is *essentially self-adjoint* and its closure yields the Hamilton operator corresponding to $\mathcal{H}^{\text{ext}}$.

The quantum analogue of the classical reduction requires restriction to the G-invariant states, i.e., to $L^2(Y, V, d\mu_Y)^G$ consisting of the G-equivariant wave functions satisfying $\mathcal{F} \circ \phi_g = \rho(g) \circ \mathcal{F}\ \forall g \in G$.

The reduced domain

$\mathcal{F} \in C^\infty(Y,V)^G$ is **uniquely** determined by its restriction to $\check{\Sigma} \subset \Sigma$, and the restricted function varies in the subspace V^K of K-invariant vectors in V, since $\mathcal{F}(q) = \mathcal{F}(k.q) = \rho(k)\mathcal{F}(q)\ \forall q \in \check{\Sigma},\ k \in K$.

This motivates to introduce the reduced domain

$$\mathrm{Fun}(\check{\Sigma}, V^K) := \{ f \in C^\infty(\check{\Sigma}, V^K) \,|\, \exists \mathcal{F} \in C_c^\infty(Y,V)^G,\ f = \mathcal{F}|_{\check{\Sigma}} \}$$

It is a pre-Hilbert space with closure $\overline{\mathrm{Fun}}(\check{\Sigma}, V^K) \simeq L^2(Y, V, d\mu_Y)^G$.

There exists a unique linear operator

$$\Delta_{\mathrm{eff}} : \mathrm{Fun}(\check{\Sigma}, V^K) \to \mathrm{Fun}(\check{\Sigma}, V^K) \quad \text{defined by the property}$$

$$\Delta_{\mathrm{eff}} f = (\Delta_Y \mathcal{F})|_{\check{\Sigma}}, \quad \text{for} \quad f = \mathcal{F}|_{\check{\Sigma}}, \quad \mathcal{F} \in C_c^\infty(Y,V)^G.$$

The ‘effective Laplace-Beltrami operator’ Δ_{eff} encodes just the restriction of Δ_Y to $C_c^\infty(Y,V)^G$.

The effective Laplace-Beltrami operator

Introduce the smooth density function $\delta : \check{\Sigma} \to \mathbb{R}_{>0}$ by

$$\delta(q) := \text{volume of the Riemannian manifold } (G.q, \eta_{G.q})$$

Choosing dual bases $\{T_\alpha\}$ and $\{T^\beta\}$ of $\mathcal{K}^\perp$, $\mathcal{B}(T_\alpha, T^\beta) = \delta_\alpha^\beta$, one has

$\delta(q) = C|\det b_{\alpha,\beta}(q)|^{\frac{1}{2}}$ with $b_{\alpha,\beta}(q) = \mathcal{B}(\mathcal{I}(q)T_\alpha, T_\beta)$ and a constant C.

Let $\Delta_{\check{\Sigma}}$ be the Laplace-Beltrami operator of $(\check{\Sigma}, \eta_{\check{\Sigma}})$.

Define $b^{\alpha,\beta}(q) := \mathcal{B}(\mathcal{I}^{-1}(q)T^\alpha, T^\beta)\ \forall q \in \check{\Sigma}$.

`The next result relies on the standard (Helgason, 72) radial-angular decomposition of` Δ_Y`, and is easily verified in local coordinates adapted to` $\check{Y} \simeq \check{\Sigma} \times G/K$.

Proposition. *On* Fun$(\check{\Sigma}, V^K)$, Δ_{eff} *takes the form*

$$\Delta_{\text{eff}} = \delta^{-\frac{1}{2}} \circ \Delta_{\check{\Sigma}} \circ \delta^{\frac{1}{2}} - \delta^{-\frac{1}{2}} \Delta_{\check{\Sigma}}(\delta^{\frac{1}{2}}) + b^{\alpha,\beta} \rho'(T_\alpha)\rho'(T_\beta)$$

where the second term is a scalar multiplication operator and the third term uses Lie algebra representation $\rho' : \mathcal{G} \to u(V)$.

The reduced quantum system

Fact 1: The complement of the dense, open submanifold $\check{Y} \subset Y$ of principal orbit type has zero measure with respect to $d\mu_Y$.

Fact 2: $d\mu_{\check{Y}} = (\delta d\mu_{\check{\Sigma}}) \times d\mu_{G/K}$ on $\check{Y} \simeq \check{\Sigma} \times G/K$ with Haar measure $d\mu_{G/K}$ on G/K and 'Riemannian measure' $d\mu_{\check{\Sigma}}$ on $(\check{\Sigma}, \eta_{\check{\Sigma}})$.

One has $\overline{\mathrm{Fun}}(\check{\Sigma}, V^K) \simeq L^2(\check{\Sigma}, V^K, \delta d\mu_{\check{\Sigma}})$, since for $\mathcal{F}_i \in C_c^\infty(Y, V)^G$

$$\int_Y (\mathcal{F}_1, \mathcal{F}_2)_V d\mu_Y = \int_{\check{Y}} (\mathcal{F}_1, \mathcal{F}_2)_V d\mu_{\check{Y}} = \int_{\check{\Sigma}} (f_1, f_2)_V \delta d\mu_{\check{\Sigma}}, \quad f_i = \mathcal{F}_i|_{\check{\Sigma}}$$

By transforming away the density δ, one gets the final result:

Theorem 2. *The reduction of the quantum system defined by the closure of $-\frac{1}{2}\Delta_Y$ on $C_c^\infty(Y, V) \subset L^2(Y, V, d\mu_Y)$ leads to the reduced Hamilton operator $-\frac{1}{2}\Delta_{\mathrm{red}}$ given by*

$$\Delta_{\mathrm{red}} = \delta^{\frac{1}{2}} \circ \Delta_{\mathrm{eff}} \circ \delta^{-\frac{1}{2}} = \Delta_{\check{\Sigma}} - \delta^{-\frac{1}{2}}(\Delta_{\check{\Sigma}} \delta^{\frac{1}{2}}) + b^{\alpha,\beta} \rho'(T_\alpha) \rho'(T_\beta).$$

Δ_{red} is essentially self-adjoint on the dense domain $\delta^{\frac{1}{2}}\,\mathrm{Fun}(\check{\Sigma}, V^K)$ in the reduced Hilbert space identified as $L^2(\check{\Sigma}, V^K, d\mu_{\check{\Sigma}})$.

Remarks on the reduced systems

• The main difference between the classical and quantum reduced Hamiltonians is the 'measure factor' $\delta^{-\frac{1}{2}}(\Delta_{\check{\Sigma}}\delta^{\frac{1}{2}})$ in the latter. This usually gives a non-trivial potential, in some cases just a constant.

• Classically, the phase space does not contain internal ('spin') degrees of freedom if $\mathcal{O}_{\text{red}} = \mathcal{O}//_0 K \simeq (\mathcal{O} \cap \mathcal{K}^{\perp})/K$ is a 1-point space. This happens with $\mathcal{O} \neq \{0\}$ only in exceptional cases. Then $\frac{1}{2}\mathcal{B}(\mathcal{I}_q^{-1}\xi, \xi) = \frac{1}{2}b^{\alpha,\beta}(q)\xi_\alpha\xi_\beta$ contributes a potential to $\mathcal{H}_{\text{red}}(q,p)$.

• Quantum mechanically, no internal degrees of freedom appear, as one gets a scalar Schrödinger operator by the reduction, if $\dim(V^K) = 1$. This happens with $\dim(V) > 1$ only in exceptional cases. Then the 'angular part' $-\frac{1}{2}b^{\alpha,\beta}\rho'(T_\alpha)\rho'(T_\beta)$ gives a potential in $-\frac{1}{2}\Delta_{\text{red}}$. These classical and quantum potential terms formally correspond upon the quantization rule $\xi_\alpha = \mathcal{B}(T_\alpha, \xi) \longrightarrow \mathrm{i}\rho'(T_\alpha)$.

• All reduced systems possess hidden $W := N_G(\Sigma)/K$ symmetry.
Results are valid also for certain pseudo-Riemannian (Y, η).
Reductions preserve integrability $\Rightarrow$ (spin) CS type models.

On restrictions of essentially self-adjoint operators

Let $A\colon \mathcal{D}(A) \to H$ be a densely defined symmetric linear operator on a Hilbert space H and $S \subset \mathcal{D}(A)$ an *invariant linear sub-manifold* of A, that is, $AS \subset S$.

Then the *restricted operator* $B := A|_S\colon S \to S$ yields a densely defined symmetric operator on the Hilbert space $\bar{S}$, where $\bar{S}$ denotes the closure of S in H. The next result is easily proven.

Lemma. *Suppose that the domain of A and the A-invariant linear sub-manifold S satisfy the additional condition*

$$P_{\bar{S}}\mathcal{D}(A) \subset S,$$

where $P_{\bar{S}}\colon H \twoheadrightarrow \bar{S}$ denotes the orthogonal projection onto the closed subspace $\bar{S}$. Then A^* is an extension *of B^*, $B^* \subset A^*$, that is, $\mathcal{D}(B^*) \subset \mathcal{D}(A^*)$ and $A^*|_{\mathcal{D}(B^*)} = B^*$.*

Consequence. *Under the above assumptions on S and $\mathcal{D}(A)$, if A is essentially self-adjoint, then so is its restriction B.*

Application to the Laplace–Beltrami operator

Fact 1: As is well-known, if (Y, η) is geodesically complete, then Δ_Y is essentially-self adjoint on the domain $C_c^\infty(Y, V) \subset L^2(Y, V, d\mu_Y)$.

Fact 2: The closure of $C_c^\infty(Y, V)^G$ is $L^2(Y, V, d\mu_Y)^G$.

Fact 3: $S := C_c^\infty(Y, V)^G$ is an invariant linear sub-manifold of Δ_Y and the condition in our lemma holds, since $\forall F \in C_c^\infty(Y, V)$

$$(P_{\bar{S}}F)(y) = \int_G \rho(g) F(g^{-1}.y) d\mu_G(g) \quad \text{and thus} \quad P_{\bar{S}}F \in S.$$

By using these facts, we can conclude that the restriction of Δ_Y to $C_c^\infty(Y, V)^G$ is an essentially self-adjoint operator of the reduced Hilbert space $L^2(Y, V, d\mu_Y)^G$.

Remark: $L^2(Y, V_\rho, d\mu_Y)^G \otimes V_{\bar{\rho}}$ can be identified with the closed subspace of $L^2(Y, d\mu_Y)$ of the 'G-symmetry type' $(\bar{\rho}, V_{\bar{\rho}})$ contragradient to (ρ, V_ρ), and the reduced Hamiltonian on $L^2(Y, V_\rho, d\mu_Y)^G$ can be obtained directly from $L^2(Y, d\mu_Y)$ as well.

Examples: Twisted spin Sutherland models

Take a compact, connected, simply connected, simple Lie group G acting on itself by **twisted conjugations** as follows:

$$\phi_g(y) := \Theta(g)yg^{-1} \quad \forall g \in G, \quad y \in Y := G \quad \text{with natural metric,}$$

where $\Theta \in \mathrm{Aut}(G)$. Symmetry reduction based on $\Theta = \mathrm{id}$ gives well-known spin Sutherland models and also the A_{N-1} spinless model with integer couplings if $G = SU(N)$ (`e.g., Etingof et al 95`).

We let $\theta := d_e\Theta$ be Dynkin diagram symmetry of $\mathcal{G}_{\mathbb{C}} \in \{A_m, D_m, E_6\}$. **Section** is provided by maximal torus T^{Θ} of fixed point set $G^{\Theta} \subset G$.

Now $\mathcal{G}_{\mathbb{C}} = \mathcal{G}_{\mathbb{C}}^{+} + \mathcal{G}_{\mathbb{C}}^{-}$ under $\theta \in \mathrm{Aut}(\mathcal{G}_{\mathbb{C}})$, and $\mathcal{G}_{\mathbb{C}}^{-}$ is irreducible module of G^{Θ} having multiplicity 1 for non-zero weights. For the Cartan subalgebra, $\mathcal{T}_{\mathbb{C}} = \mathcal{T}_{\mathbb{C}}^{+} + \mathcal{T}_{\mathbb{C}}^{-}$. Introduce notation

$\Delta = \{\alpha\}$: roots of $(\mathcal{T}_{\mathbb{C}}^{+}, \mathcal{G}_{\mathbb{C}}^{+})$ with associated roots vectors X_{α}^{+}

$\Gamma = \{\lambda\}$: non-zero weights of $(\mathcal{T}_{\mathbb{C}}^{+}, \mathcal{G}_{\mathbb{C}}^{-})$ with weight vectors X_{λ}^{-}

Next describe result of quantum reduction; classical case at `RAQIS05`.

Roots and weights for involutive diagram automorphisms

If $\mathcal{G}_\mathbb{C} = D_{n+1}$, then $\mathcal{G}_\mathbb{C}^+ = B_n$ and $\mathcal{G}_\mathbb{C}^-$ spans its defining irrep:

$$\Delta_+ = \{e_k \pm e_l,\, e_m \,|\, 1 \le k < l \le n,\ 1 \le m \le n\},$$

$$\Gamma_+ = \{e_m \,|\, 1 \le m \le n\}. \qquad \text{One may take}$$

$$\mathcal{T}_\mathbb{C}^+ \ni q = \mathrm{diag}(q_1, \ldots, q_n, 0, 0, -q_n, \ldots, -q_1) \quad \text{and} \quad e_m : q \mapsto q_m$$

If $\mathcal{G}_\mathbb{C} = A_{2n-1}$, then $\mathcal{G}_\mathbb{C}^+ = C_n$ with $\Gamma_+ = \{e_k \pm e_l \,|\, 1 \le k < l \le n\}$
and $\Delta_+ = \{e_k \pm e_l,\, 2e_m \,|\, 1 \le k < l \le n,\ 1 \le m \le n\}$.
Now $\mathcal{T}_\mathbb{C}^+ \ni q = \mathrm{diag}(q_1, \ldots, q_n, -q_n, \ldots, -q_1)$ and $e_m : q \mapsto q_m$

For the 'richest case' $\mathcal{G}_\mathbb{C} = A_{2n}$ one has $\mathcal{G}_\mathbb{C}^+ = B_n$ and

$$\Gamma_+ = \{e_k \pm e_l,\, e_m,\, 2e_m \,|\, 1 \le k < l \le n,\ 1 \le m \le n\}.$$

$$\mathcal{T}_\mathbb{C}^+ \ni q = \mathrm{diag}(q_1, \ldots, q_n, 0, -q_n, \ldots, -q_1) \quad \text{and} \quad e_m : q \mapsto q_m$$

Reduced Hamiltonian and spectrum

Parametrize reduced configuration space $\check{T}^{\Theta}$ by $e^{\mathrm{i}q}$, and choose orthonormal basis $\{\mathrm{i}K_j^-\}$ of $\mathcal{T}^-$. Define $\varrho^\theta := \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha + \frac{1}{2}\sum_{\lambda\in\Gamma_+}\lambda$.

The symmetry reduction of the Laplace–Beltrami operator Δ_G on G associated with a unitary representation (ρ, V_ρ) of G is given by

$$\Delta_{\mathrm{red}} = \Delta_{\check{T}^\Theta} + \langle \varrho^\theta, \varrho^\theta \rangle - \frac{1}{4}\sum_{\alpha\in\Delta} \frac{\rho'(X_\alpha^+)\rho'(X_{-\alpha}^+)}{\sin^2\left(\frac{\alpha(q)}{2}\right)}$$

$$-\frac{1}{4}\sum_{\lambda\in\Gamma} \frac{\rho'(X_\lambda^-)\rho'(X_{-\lambda}^-)}{\cos^2\left(\frac{\lambda(q)}{2}\right)} + \frac{1}{4}\sum_j \rho'(\mathrm{i}K_j^-)^2$$

Δ_{red} acts on reduced Hilbert space $L^2(\check{T}^\Theta, V_\rho^{\mathrm{inv}}, d\mu_{\check{T}^\Theta})$, where V_ρ^{inv} contains the T^Θ singlets in V_ρ. Since the reduced Hilbert space is naturally identical to the space of G-singlets

$$\left(L^2(G, d\mu_G)\otimes V_\rho\right)^G, \quad L^2(G, d\mu_G) = \oplus_{\Lambda\in L^+} V_{(\Lambda\circ\theta)^*}\otimes V_\Lambda \quad \text{under } G,$$

and thus the spectrum of Δ_G is known (L^+: highest weights), the diagonalization of Δ_{red} becomes a Clebsch-Gordan problem.

Explicit spectra in some cases for $G = SU(N)$

Label representation ρ of G by highest weight ν, denote it as V_ν. Then the eigenvalues of Δ_{red} are of the form $-\langle \Lambda + 2\varrho, \Lambda \rangle$, where Λ runs over the admissible highest weights, for which

$$\dim \left(V_{(\Lambda\circ\theta)^*} \otimes V_\Lambda \otimes V_\nu \right)^G = N_{\Lambda,\nu}^{\Lambda\circ\theta} \neq 0.$$

This can be solved **explicitly** if $G = SU(N)$ and $\nu = k\Lambda_1$ with fundamental weight Λ_1: $V_\nu = \mathcal{S}^k \left(\mathbb{C}^N \right)$. For $\theta = \text{id}$ (`Etingof et al`)

$$\Lambda = \lambda + c\varrho, \quad \forall \lambda \in L^+_{SU(N)}, \quad k = cN \quad (c \in \mathbb{Z}_+)$$

and $\dim(V^{\text{inv}}_{cN\Lambda_1}) = 1$. Recovers spinless Sutherland spectrum for the integral couplings, $g = (c+1)$, which admit hidden G symmetry.

If θ is non-trivial, then $\Lambda^* = \Lambda \circ \theta$ and we find $\Lambda = \lambda + \sum_{i=1}^{N-1} c_{i+1} \Lambda_i$, where λ is an arbitrary self-conjugate highest weight of $SU(N)$, the Λ_i are the fundamental weights and

$\mathbf{c} = (c_1, c_2, \ldots, c_N) \in \mathbb{Z}_+^N$ with $c_{N+1-a} = c_a$, $\sum_{a=1}^N c_a = k$.

For any given k, the number of solutions for $\mathbf{c}$ equals $\dim(V^{\text{inv}}_{k\Lambda_1}) > 0$.

On some examples of spin Sutherland type models containing the **spinless** BC_n models with 3 coupling constants

$\mathcal{R}$: crystallographic root system

$$\mathcal{H}_{\mathcal{R}}(q,p) := \frac{1}{2}\langle p,p\rangle + \sum_{\alpha\in\mathcal{R}_+} \frac{g_\alpha^2}{\sinh^2 \alpha(q)}$$

This defines Sutherland type model for any root system [OP, 76]. Coupling constants g_α^2 may arbitrarily **depend on orbits** of the corresponding reflection group. An important case is $\mathcal{R} = BC_n$:

$$\mathcal{H}_{BC_n} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \sum_{1\leq j<k\leq n}\Big(\frac{g^2}{\sinh^2(q^j - q^k)} + \frac{g^2}{\sinh^2(q^j + q^k)}\Big) + \sum_{k=1}^{n}\Big(\frac{g_1^2}{\sinh^2(q^k)} + \frac{g_2^2}{\sinh^2(2q^k)}\Big)$$

[OP, 76]: BC_n model is 'projection' of geodesics on symmetric space $SU(n+1,n)/(S(U(n+1)\times U(n)))$ **if** $g_1^2 - 2g^2 + \sqrt{2}gg_2 = 0$.
Why this symmetric space? Can one get rid of the restriction in the classical Hamiltonian reduction framework? `(We answered these questions in arXiv:math-ph/0604073 and in arXiv:math-ph/0609085.)`

Preliminaries for reduction of motion on group G

Take a non-compact, connected, real simple Lie group G with finite center and denote by G_+ its maximal compact subgroup. Equip G with the pseudo-Riemannian structure induced by the Killing form $\langle\ ,\ \rangle$ of $\mathcal{G}$. We describe the reduction of free motion on G at **any** value of the momentum map for 'left $\times$ right' action of $G_+ \times G_+$.

Consider $\mathcal{G}_+ := \mathrm{Lie}(G_+)$ and Cartan decomposition $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$. Choose maximal Abelian subspace $\mathcal{A} \subset \mathcal{G}_-$. Centralizer

$$\mathcal{M} := \{Z \in \mathcal{G}_+ \,|\, [Z, X] = 0 \ \ \forall X \in \mathcal{A}\} = \mathrm{Lie}(M) \quad \text{with}$$

$$M := \{m \in G_+ \,|\, m X m^{-1} = X \quad \forall X \in \mathcal{A}\} \quad \text{using matrix notations}$$

$M_{\mathrm{diag}} \subset G_+ \times G_+$ principal isotropy group for $G_+ \times G_+$ action on G. Flat section is provided by $A := \exp(\mathcal{A}) = \{e^q \,|\, q \in \mathcal{A}\}$. One has

$$\mathcal{G}_- = \mathcal{A} + \mathcal{A}^\perp,\ \mathcal{G}_+ = \mathcal{M} + \mathcal{M}^\perp,\ (\mathcal{A}^\perp + \mathcal{M}^\perp) = \sum_{\alpha \in \mathcal{R}} \mathcal{G}_\alpha,\ m_\alpha := \dim(\mathcal{G}_\alpha)$$

$\mathcal{G}_\alpha$ is joint eigensubspace for ad_q, $q \in \mathcal{A}$ and $\alpha \in \mathcal{R}$ is restricted root.

Reduced systems from $G_+ \times G_+$ action on G

Now $\mathcal{O}_{\rm red} = (\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}^\perp_{\rm diag}/M_{\rm diag}$ with orbit $\mathcal{O}^l \oplus \mathcal{O}^r$ of $G_+ \times G_+$. Decomposing $(\xi^l, \xi^r) \in \mathcal{O}$ as $\xi^{l,r} = \xi^{l,r}_{\mathcal{M}} + \xi^{l,r}_{\mathcal{M}^\perp}$, $\mathcal{H}_{\rm red}$ is the following $M_{\rm diag}$-invariant function on $T^*\check{A} \times (\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}^\perp_{\rm diag}$:

$$2\,\mathcal{H}_{\rm red}(q,p,\xi^l,\xi^r) = \langle p,p\rangle + \langle \xi^l_{\mathcal{M}}, \xi^l_{\mathcal{M}}\rangle - \langle \xi^l_{\mathcal{M}^\perp}, w^2(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}\rangle$$
$$-\langle \xi^r_{\mathcal{M}^\perp}, w^2(\mathrm{ad}_q)\xi^r_{\mathcal{M}^\perp}\rangle + \langle \xi^r_{\mathcal{M}^\perp}, w^2(\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}\rangle - \langle \xi^r_{\mathcal{M}^\perp}, w^2(\tfrac{1}{2}\mathrm{ad}_q)\xi^l_{\mathcal{M}^\perp}\rangle$$

with $w(z) = \frac{1}{\sinh z}$, $\xi^l_{\mathcal{M}} + \xi^r_{\mathcal{M}} = 0$. Spin Sutherland model in general.

One has the density $\delta(e^q) = \prod_{\alpha\in\mathcal{R}_+} |\sinh(\alpha(q))|^{m_\alpha}$. As calculated by Olshanetsky and Perelomov (1978), this gives rise to the potential

$$\frac{1}{2}\delta^{-\frac{1}{2}}\Delta(\delta^{\frac{1}{2}}) = \frac{1}{2}\langle \varrho,\varrho\rangle + \sum_{\alpha\in\mathcal{R}_+} \frac{m_\alpha}{4}\left(\frac{m_\alpha}{2} + m_{2\alpha} - 1\right)\frac{\langle\alpha,\alpha\rangle}{\sinh^2(\alpha(q))}$$

where $\varrho := \frac{1}{2}\sum_{\alpha\in\mathcal{R}_+} m_\alpha \alpha$ and $m_{2\alpha} \neq 0$ only for $\alpha = e_j \in BC_n$. Similar result holds if G compact and G_+ fixed by involution of G.

How to obtain spinless models?

The basic example and the 'KKS mechanism'

Consider $G := SL(n,\mathbb{C})$ with Cartan involution $\Theta : g \mapsto (g^\dagger)^{-1}$.
$\mathcal{T}_{n-1}$: Lie algebra of maximal torus $\mathbf{T}_{n-1} \subset SU(n) = G_+$.
Now $sl(n,\mathbb{C}) = su(n) + \mathrm{i}\, su(n)$ and $\mathcal{A} = \mathrm{i}\mathcal{T}_{n-1}$, $M = \mathbf{T}_{n-1}$.

$$\text{If } \mathcal{O}^r = \{0\}, \text{ then } \mathcal{O}_{\text{red}} \simeq (\mathcal{O}^l \cap \mathcal{T}_{n-1}^\perp)/\mathbf{T}_{n-1}.$$

This is 1-point space iff $\mathcal{O}^l$ is minimal orbit of $SU(n)$.

The minimal orbits of $SU(n)$ are $\mathcal{O}_{n,\kappa,\pm}$ for $\kappa > 0$, consisting of the elements $\xi = \pm\mathrm{i}\Big(uu^\dagger - \frac{u^\dagger u}{n}\mathbf{1}_n\Big)$ for some $u \in \mathbb{C}^n$, $u^\dagger u = n\kappa$. Imposing $\xi_{a,a} = 0$ requires $u_a = \sqrt{\kappa}e^{\mathrm{i}\beta_a}$, leading to representative with $\xi_{a,b} = \pm\mathrm{i}\kappa(1 - \delta_{a,b})$. Reproduces original Sutherland model (as shown by Kazhdan-Kostant-Sternberg in 78).

'KKS mechanism':
In addition to starting with 1-point orbits, **one gets 1-point space for $\mathcal{O}_{\text{red}}$ if G_+ has an $SU(k)$ factor and above arguments are applicable to $\mathcal{O}_{\text{red}} = (\mathcal{O}^l \oplus \mathcal{O}^r) \cap \mathcal{M}_{\text{diag}}^\perp / M_{\text{diag}}$.**

Deformation of (spin) Sutherland models using characters

Suppose that $C \in \mathcal{G}_+ \simeq \mathcal{G}_+^*$ forms a 1-point coadjoint orbit of G_+.

/Such character exists iff G/G_+ is Hermitian symmetric space./

Then $(\mathcal{O}^r + yC)$ and $(\mathcal{O}^l - yC)$ 1-parameter families of G_+ orbits, and the constraints are not affected by the value of y.

$$\mathcal{O}^y_{\text{red}} := \Big(\big(\mathcal{O}^l - yC\big) \oplus \big(\mathcal{O}^r + yC\big)\Big) \cap \mathcal{M}^{\perp}_{\text{diag}} / M_{\text{diag}}, \quad \forall y \in \mathbb{R}$$

yields deformation of system associated with $y = 0$.

If $\mathcal{O}^{y=0}_{\text{red}}$ is a 1-point space, then this holds $\forall y \in \mathbb{R}$.

Besides $G = SL(n, \mathbb{C})$, the KKS mechanism works **iff** $G = SU(m, n)$. In this case $G_+ = S(U(m) \times U(n)) = SU(m) \times SU(n) \times U(1)$ and a 1-parameter family of characters exists.

Some details on $G = SU(m,n)$, $m \geq n$

$$SU(m,n) = \{g \in SL(m+n,\mathbb{C}) \,|\, g^\dagger I_{m,n} g = I_{m,n}\}$$
$$su(m,n) = \{X \in \; sl(m+n,\mathbb{C}) \,|\, X^\dagger I_{m,n} + I_{m,n} X = 0\}$$

where $I_{m,n} := \mathrm{diag}(\mathbf{1}_m, -\mathbf{1}_n)$. Any $X \in \mathcal{G} = su(m,n)$ has the form

$$X = \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix}$$

with $B \in \mathbb{C}^{m\times n}$, $A \in u(m)$, $D \in u(n)$ and $\mathrm{tr}\, A + \mathrm{tr}\, D = 0$. With Cartan involution $\Theta : g \mapsto (g^\dagger)^{-1}$, $\theta : X \mapsto -X^\dagger$, one obtains $G_+ = S(U(m)\times U(n))$ and $\mathcal{G}_+ = su(m) \oplus su(n) \oplus \mathbb{R}C_{m,n}$. Then $\mathcal{G}_-$ consists of block off-diagonal, hermitian matrices. Next we fix maximal Abelian subspace $\mathcal{A} \subset \mathcal{G}_-$ and describe its centralizer.

$$\mathcal{A} := \left\{ q := \begin{pmatrix} \mathbf{0}_n & 0 & Q \\ 0 & \mathbf{0}_{m-n} & 0 \\ Q & 0 & \mathbf{0}_n \end{pmatrix} \,\middle|\, Q = \mathrm{diag}(q^1, \ldots, q^n),\ q^j \in \mathbb{R} \right\}$$

Using $\chi := \mathrm{diag}(\chi_1, \ldots, \chi_n)\ \forall \chi_j \in \mathbb{R}$, centralizer of $\mathcal{A}$ reads

$$\mathcal{M} = \{\mathrm{diag}(\mathrm{i}\chi, \gamma, \mathrm{i}\chi) \mid \gamma \in u(m-n),\ \mathrm{tr}\,\gamma + 2\mathrm{i}\mathrm{tr}\,\chi = 0\} \subset \mathcal{G}_+$$

$$M = \{\mathrm{diag}(e^{\mathrm{i}\chi}, \Gamma, e^{\mathrm{i}\chi}) \mid \Gamma \in U(m-n),\ (\det \Gamma)(\det e^{\mathrm{i}2\chi}) = 1\} \subset G_+.$$

Define $e_k \in \mathcal{A}^*$ $(k = 1, \ldots, n)$ by $e_k(q) := q^k$. **Restricted roots:**

BC_n : $\mathcal{R}_+ = \{e_j \pm e_k\,(1 \leq j < k \leq n),\ 2e_k, e_k\,(1 \leq k \leq n)\}$ if $m > n$

C_n : $\mathcal{R}_+ = \{e_j \pm e_k\,(1 \leq j < k \leq n),\ 2e_k\,(1 \leq k \leq n)\}$ if $m = n$

multiplicities: $m_{e_j \pm e_k} = 2, \quad m_{2e_k} = 1, \quad m_{e_k} = 2(m-n)$

For $G = SU(m,n)$, the system of restricted roots is of BC_n type if $m > n$ and of C_n type if $m = n$. The 1-parameter family of characters is spanned by $C_{m,n} := \mathrm{diag}(\mathrm{i}n\mathbf{1}_m, -\mathrm{i}m\mathbf{1}_n)$.

Spinless BC_n Sutherland models result in the following cases.

• If $m = n$: $\mathcal{O}^l := \mathcal{O}_{n,\kappa,+} + \{xC_{n,n}\}$, $\mathcal{O}^r := \{yC_{n,n}\}$, $\forall x, y, \kappa$.
One gets 3 couplings $g^2 = \kappa^2/4$, $g_1^2 = xyn^2/2$, $g_2^2 = (x-y)^2 n^2/2$.

• If $m = n+1$: one obtains the BC_n model by taking
$\mathcal{O}^l := \mathcal{O}_{n+1,\kappa,+} + \{xC_{n+1,n}\}$, $\mathcal{O}^r := \{yC_{n+1,n}\}$ with
3 parameters subject to $\kappa + x + y \geq 0$ and $\kappa - n(x+y) \geq 0$.

• If $m \geq n+1$: model with 2 independent couplings comes from
$\mathcal{O}^l = \mathcal{O}_{n,\kappa,+} + \{xC_{m,n}\}$ and $\mathcal{O}^r = \{yC_{m,n}\}$ with $x = -y$.

`A. Oblomkov (math.RT/0202076)` considered **quantum** Hamiltonian reduction for holomorphic analogue of the above $SU(n,n)$ case.

FINAL REMARKS

One recovers the `Olshanetsky-Perelomov` (1976) and probably also the `Inozemtsev-Meshcheryakov` (1985) Lax pairs of the BC_n Sutherland model using the reductions based on $SU(n+1,1)$ and $SU(n,n)$. It could be interesting to find corresponding dynamical r-matrices.

Construction can be applied also to compact simple Lie groups. This amounts to replacing $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$ by $\mathcal{G}_{\text{compact}} = \mathcal{G}_+ + \mathrm{i}\mathcal{G}_-$, and leads to trigonometric version of (spin) Sutherland models.

May replace symmetry group $G_+ \times G_+$ by other groups $G'_+ \times G''_+$. Results survive if a dense subset of G (of principal orbit type) admits parametrization as $g = g'_+ e^q g''_+$. This works for fixpoint sets of (commuting) involutions, i.e., for the hyperpolar 'Hermann actions'.

We plan to explore the family of systems, both classically and quantum mechanically, that can be associated with hyperpolar actions on symmetric spaces and on underlying Lie groups.