16.901 Home Assignment 1: ODE Integration
Simulation of Lateral Motion of an Airplane
Part 1 Due: Part 2 Due:

1 Background

Modeling the dynamics of an airplane can be very complex. Twelve state variables are required to describe the motion: three states for position, three for velocity, three for angluar orientation, and three angular rates. However, this system of twelve coupled differential equations can be simplified greatly by linearization and some assumptions. These simplifications yield two decoupled four-state systems, one describing longitudinal motion and the other describing lateral motion.

The longitudinal modes can be simplified even more to give fairly accurate decoupled models of the familiar pheugoid and short-period modes. The lateral system cannot be simplified as easily to solve for the three lateral modes: Dutch roll, spiral mode, and roll mode. The Dutch roll mode is a combination of yawing and rolling oscillations. In the roll mode, the roll rate reaches steady state quickly. The spiral mode can be slightly stable or unstable. An unstable spiral mode can lead to divergence from the flight path or result in a spiral dive. These lateral modes will be the topic of this assignment.

1.1 Definition of axes

This is the body coordinate frame. The motion of the aircraft is measured relative to a fixed frame making the states the relative position, motion, angular orientation, and angular rate of the body frame relative to the fixed frame.

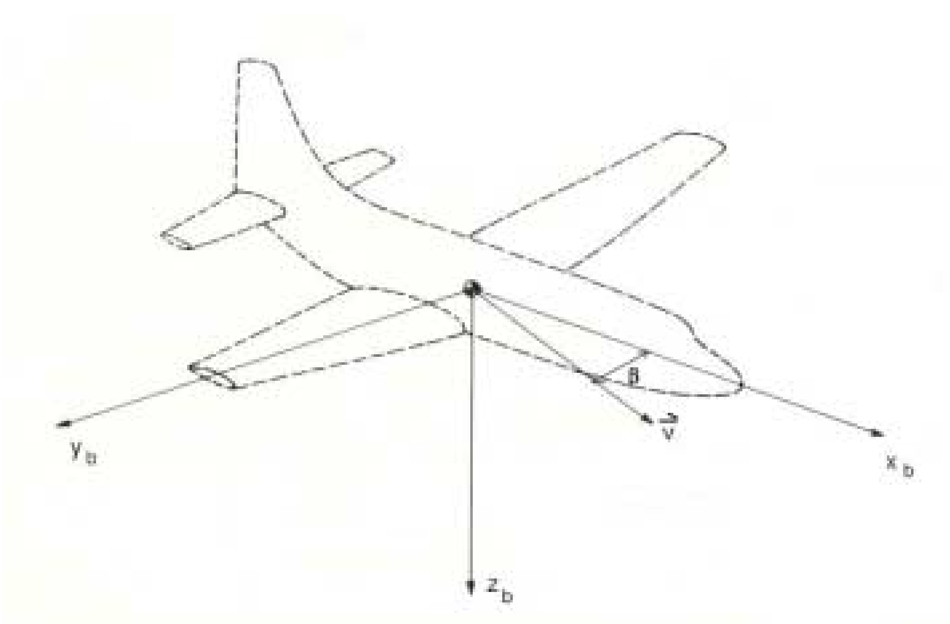

Figure 1: Aircraft Dynamics Coordinate System

1.2 Definition of variables

The following table introduces the relevant variables.

	Symbol	Description
States	$\Delta\beta$	Side-slip angle perturbation
	Δp	Roll rate perturbation
	Δr	Yaw rate perturbation
	$\Delta\phi$	Roll angle perturbation
Forces and Moments	Y	Side force (Force in y direction)
	L	Rolling moment (Moment about x axis)
	N	Yawing moment (Moment about z axis)
Normalizing quantities	Q	Dynamic pressure
	S	Planform area
	b	Wingspan
	m	Mass of aircraft
	I_i	Moment of inertia about i axis
Misc.	g	Gravitational acceleration
	u_0	Velocity in x direction
	θ_0	Pitch angle

Table 1: Explanation of Variables

The linearized equations of motion are written in terms of stability derivates, the first derivatives of the forces and moments with respect to the states and inputs. These are generally written in subscript notation. For example, $Y_\beta \equiv \frac{\partial Y}{\partial \beta}$. Often stability derivatives are given in a non-dimensional form, such as C_{y_β} as the non-dimensionalized form of Y_β. Also, note that a zero subscript denotes a state value that the equations of motion were linearizeed about.

1.3 Equations of motion

The linearized equations of lateral motion are given below.

$$\frac{d}{dt}\Delta\beta = \frac{QS}{mu_0}C_{y_\beta}\Delta\beta + C_{y_p}\frac{QSb}{2mu_0^2}\Delta p + \left(C_{y_r}\frac{QSb}{2mu_0^2} - 1\right)\Delta r + \frac{g\cos\theta_0}{u_0}\Delta\phi$$

$$\frac{d}{dt}\Delta p = C_{l_\beta}\frac{QSb}{I_x}\Delta\beta + C_{L_p}\frac{QSb^2}{2I_xu_0}\Delta p + C_{l_r}\frac{QSb^2}{2I_xu_0}\Delta r$$

$$\frac{d}{dt}\Delta r = C_{n_\beta}\frac{QSb}{I_z}\Delta\beta + C_{n_p}\frac{QSb^2}{2I_zu_0}\Delta p + C_{n_r}\frac{QSb^2}{2I_zu_0}\Delta r$$

$$\frac{d}{dt}\Delta\phi = \Delta p$$

Values of the stability derivatives and dimensions for selected airplanes are given in the following tables.

Quantity	Values for 747	Values for F-104
C_{y_β}	-0.96	-1.17
C_{y_p}	0	0
C_{y_r}	0	0
C_{l_β}	-0.221	-0.175
C_{l_p}	-0.45	-0.285
C_{l_r}	0.101	0.265
C_{n_β}	0.150	0.50
C_{n_p}	-0.121	-0.14
C_{n_r}	-0.30	-0.75
S	$5500\ ft^2$	$196.1\ ft^2$
b	$195.68\ ft$	$21.94\ ft$
u_0	$286.6\ \frac{ft}{s}$	$294.64\ \frac{ft}{s}$
m	$19770\ slug$	$506.2\ slug$
I_x	$18.2 * 10^6\ slug\ ft^2$	$3549\ slug\ ft^2$
I_z	$49.7 * 10^6\ slug\ ft^2$	$59669\ slug\ ft^2$

Table 2: Stability derivatives and dimensions for 747 and F-104 at sea level

2 Assignment Part 1 - Due:

2.1 Analytic solution

Do the following:

- Solve for the eigenvalues and eigenvectors for both airplanes. Assume level flight. State which eigenvalues and eigenvectors correspond to each lateral mode. Interpret what the eigenvalues and eigenvectors show about the dynamic behavior of each aircraft for each mode. Specifically, comment on what the eigenvectors say about the the relative magnitudes and phase differences of the states.

- Graph the analytical solution of the response from of each airplane $t = 0s$ to $t = 100s$ for the following sets of initial conditions:

$$\Delta\beta(0) = 0.1\,rad, \Delta p(0) = 0, \Delta r(0) = 0, \Delta\phi(0) = 0$$

$$\Delta\beta(0) = 0, \Delta p(0) = 0, \Delta r(0) = 0, \Delta\phi(0) = 0.1\,rad$$

  Relate the information obtained from the eigenvalues and eigenvectors to the results observed in the analytic solutions.

2.2 Analysis of numerical integration methods

Do the following:

- Find the maximum time step that can capture the physics of the problem for the following methods: First order Adams-Bashfourth, first order Adams-Moulton, second order Runge-Kutta. Plot the stability regions for each method and overlay the eigenvalues.
- In the case of first order Adams-Bashfourth (Forward Euler), find the time for the divergent numerical solution to double if a time step twice the maximum is used.
- Using the results from above, determine which method method will be the most computationally efficient. Include in your answer considerations about accuracy and stability. Also, explain how the specifics of this problem play a role in the selection of the best numerical method.

3 Assignment Part 2 - Due:

3.1 Implementation of numerical methods

Do the following:

- Implement the following numerical methods to integrate the system of equations : First and second order Adams-Bashfourth, first and second order Adams-Moulton, second and fourth order Runge-Kutta. Give a brief description of each method and how you implemented it. Append the commented source code to your assignment.
- Compare the solution generated by the six implemented numerical methods using a time step of $dt = 0.3s$ on the interval $t = 0s$ to $t = 10s$ for the initial conditions:

$$\Delta\beta\left(0\right) = 0.1\,rad, \Delta p\left(0\right) = 0.1\,\frac{rad}{s}, \Delta r\left(0\right) = 0.1\,\frac{rad}{s}, \Delta\phi\left(0\right) = 0.1\,rad$$

3.2 Study of numerical accuracy and computational work

Do the following:

- For this assignment, the measure of accuracy will be the maximum error percentage of the states at the end of the simulation interval. Find the maximum time step for each method such that the error is within $\pm 1\%$.
- In order to compare the work involved in these methods, we need a single 'currency' which is relatively independant of the particular computer, network, version of software compiler, etc. We will use the computational time required to run a single iteration of the forward Euler method as our Work Unit (WU). To find the WU's for an iteration of the other methods, run each of them for a few hundred iterations and calculate the average time required for a single iteration. MAKE SURE TO DO THIS ON THE SAME COMPUTER, IMMEDIATELY AFTER EACH OTHER to reduce the possibility of varying conditions affecting the timings. Then, normalize these timings by the forward Euler timings to calculate the WU for an iteration of each method. Report the WU for an iteration of each method in a table.
- For this assignment, the measure of accuracy will be the maximum error percentage of the states compared with the analyical at the end of the simulation interval. Find the maximum

time step for each method such that the error is within $\pm 1\%$. Use the following initial conditions:

$$\Delta\beta\,(0) = 0.1\,rad, \Delta p\,(0) = 0.1\,\frac{rad}{s}, \Delta r\,(0) = 0.1\,\frac{rad}{s}, \Delta\phi\,(0) = 0.1\,rad$$

- Thoroughly discuss the results of the accuracy and work study of the different integration methods. Try to explain the WU results based on your understanding of the methods and their accuracy and stability.