Integrable Hamiltonian systems from reductions of doubles of compact Lie groups II

Generalizations of spin Sutherland models

László Fehér, University of Szeged and Wigner RCP, Budapest

Yesterday, we derived the spin Sutherland model with 'collective spin variables',

$$H_{\text{spin-Suth}}(q,p,\xi) = \frac{1}{2}\langle p,p\rangle + \frac{1}{8}\sum_{\alpha\in\mathfrak{R}} \frac{2}{|\alpha|^2}\frac{|\xi_\alpha|^2}{\sin^2(\alpha(q)/2)},$$

from reduction of 'geodesic motion' on the cotangent bundle T^*G of a compact Lie group G and discussed its integrability.

We also mentioned the Gibbons–Hermsen (1984) model

$$H_{\text{G-H}} = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \frac{1}{8}\sum_{j\neq k}\frac{|(S_j S_k^\dagger)|^2}{\sin^2((q_j-q_k)/2)},$$

for which a complex row vector $S_j := [S_{j1},\dots,S_{jd}] \in \mathbb{C}^d$, $d\geq 2$, is attached to the particle with coordinate q_j, representing internal degrees of freedom.

In today's talk, we focus on generalizations of the first kind of models, utilizing Heisenberg models instead of cotangent bundles.

If time permits, we shall also outline a generalization of the Gibbons–Hermsen model (the trigonometric real form of the spin Ruijsenaars–Schneider model of Krichever and Zabrodin (1995)).

This lecture is based on the following papers:

- LF, *Poisson–Lie analogues of spin Sutherland models*, Nucl. Phys. B 949, 114807 (2019); arXiv:1809.01529

- M. Fairon, LF and I. Marshall, *Trigonometric real form of the spin RS model of Krichever and Zabrodin*, Ann. Henri Poincaré 22, 615-675 (2021); arXiv:2007.09658

- LF, *Poisson reductions of master integrable systems on doubles of compact Lie groups*, Ann. Henri Poincaré 24, 1823-1876 (2023); arXiv:2302.14392

- LF, *Poisson–Lie analogues of spin Sutherland models revisited*, J. Phys A: Math. Theor. 57, 205202 (2024); arXiv:2402.02990

Reminder on the notion of integrability

Definition I. Let $(\mathcal{M}, P_{\mathcal{M}})$ be a finite dimensional, connected, C^∞ Poisson manifold, and $\mathfrak{H}$ an Abelian Poisson subalgebra of $C^\infty(\mathcal{M})$ subject to the conditions:

1. As a commutative algebra of functions $\mathfrak{H}$ has functional dimension $\mathrm{ddim}(\mathfrak{H}) = \ell$.
2. The Hamiltonian vector fields of the elements of $\mathfrak{H}$ are complete and span an ℓ dimensional subspace of the tangent space over a dense open subset of $\mathcal{M}$.
3. The commutant $\mathfrak{F}$ of $\mathfrak{H}$ in $C^\infty(\mathcal{M})$, which contains the joint constants of motion of the Hamiltonians $\mathcal{H} \in \mathfrak{H}$, has functional dimension $\mathrm{ddim}(\mathfrak{F}) = \mathrm{dim}(\mathcal{M}) - \ell$.

We refer to the quadruple $(\mathcal{M}, P_{\mathcal{M}}, \mathfrak{H}, \mathfrak{F})$, or simply $\mathfrak{H}$, as a *(degenerate) integrable system of rank* ℓ. The standard notion of Liouville integrability results if $\mathcal{M}$ is a symplectic manifold and $\ell = \mathrm{dim}(\mathcal{M})/2$. Liouville integrability on Poisson manifolds is the case for which $\ell = k$, where k is half the dimension of the maximal symplectic leaves. When $\ell < k$, both on Poisson and symplectic manifolds, then one obtains the case of degenerate integrability, alternatively called superintegrability. A single Hamiltonian is called (super)integrable if it is a member of $\mathfrak{H}$ obeying the definition.

Reminder on the general picture

Let G be a (connected and simply connected) compact Lie group with simple Lie algebra $\mathcal{G}$. Denote $\mathcal{G}^{\mathbb{C}}$ and $G^{\mathbb{C}}$ the complexifications, and define $\mathfrak{P} := \exp(\mathrm{i}\mathcal{G}) \subset G^{\mathbb{C}}$. Example: $G = SU(n)$, $G^{\mathbb{C}} = SL(n, \mathbb{C})$, $\mathfrak{P} = \{X \in SL(n, \mathbb{C}) \mid X^{\dagger} = X,\ X \text{ positive}\}$.

One has the following 3 'classical doubles' of G:

Cotangent bundle $\quad T^*G \simeq G \times \mathcal{G}^* \simeq G \times \mathcal{G} =: \mathcal{M}_1$

Heisenberg double $\quad G^{\mathbb{C}}_{\mathbb{R}} \simeq G \times G^* \simeq G \times \mathfrak{P} =: \mathcal{M}_2$

Internally fused quasi-Poisson double $\quad G \times G =: \mathcal{M}_3$

The pull-backs of the relevant rings of invariants

$$C^{\infty}(G)^G, \quad C^{\infty}(\mathcal{G})^G, \quad C^{\infty}(\mathfrak{P})^G$$

give rise to two 'master integrable systems' on each double.

The group G acts on these phase spaces by 'diagonal conjugations', i.e., by the diffeomorphisms

$$A^i_\eta : (x, y) \mapsto (\eta x \eta^{-1}, \eta y \eta^{-1}), \quad \forall (x, y) \in \mathcal{M}_i \ (i = 1, 2, 3),\ \eta \in G.$$

The G-invariant functions form closed Poisson algebras, and thus the quotient space $\mathcal{M}_i^{\mathrm{red}} \equiv \mathcal{M}_i/G$ becomes a (singular) Poisson space, which carries the corresponding reduced integrable systems.

Plan of the lecture

- Integrable ‘master system’ on the Heisenberg double
- Poisson reduction of the master system: reduced integrability
- Two descriptions of the reduced Poisson brackets
- Connection to the spin Sutherland models
- The dual system in a nutshell
- Conclusion
- (Appendix: On a generalization of the Gibbons–Hermsen model)

Preparations. Fix a maximal Abelian subalgebra, $\mathcal{G}_0 < \mathcal{G}$. A choice of positive roots with respect to the Cartan subalgebra $\mathcal{G}_0^{\mathbb{C}} < \mathcal{G}^{\mathbb{C}}$ leads to the triangular decomposition

$$\mathcal{G}^{\mathbb{C}} = \mathcal{G}_-^{\mathbb{C}} + \mathcal{G}_0^{\mathbb{C}} + \mathcal{G}_+^{\mathbb{C}}.$$

Equip the realification $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ of $\mathcal{G}^{\mathbb{C}}$ with bilinear form $\langle X, Y\rangle_{\mathbb{I}} := \Im\langle X, Y\rangle$, where $\langle -, -\rangle$ is the Killing form of $\mathcal{G}^{\mathbb{C}}$. Then one obtains the decomposition (a Manin triple)

$$\mathcal{G}_{\mathbb{R}}^{\mathbb{C}} = \mathcal{G} + \mathcal{B} \quad \text{with} \quad \mathcal{B} := \mathrm{i}\mathcal{G}_0 + \mathcal{G}_+^{\mathbb{C}} =: \mathcal{B}_0 + \mathcal{B}_+.$$

Let $G_{\mathbb{R}}^{\mathbb{C}}$ a connected and simply connected Lie group with Lie algebra $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$. We may write any $X \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ as $X = X_{\mathcal{G}} + X_{\mathcal{B}}$ or as $X = X_+ + X_0 + X_-$ or as $X = Y_1 + \mathrm{i}Y_2$ $(Y_1, Y_2 \in \mathcal{G})$. The complex conjugation θ with respect to $\mathcal{G}$ is a Cartan involution and it lifts to the involution Θ of $G_{\mathbb{R}}^{\mathbb{C}}$. We have the anti-automorphisms

$$Z \mapsto Z^\dagger := -\theta(Z), \qquad K \mapsto K^\dagger := \Theta(K^{-1}), \qquad \forall Z \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}},\ \forall K \in G_{\mathbb{R}}^{\mathbb{C}}.$$

By using the subgroups $G < G_{\mathbb{R}}^{\mathbb{C}}$ and $B := \exp(\mathcal{B}) < G_{\mathbb{R}}^{\mathbb{C}}$, every element $K \in G_{\mathbb{R}}^{\mathbb{C}}$ admits the unique (Iwasawa) decompositions:

$$K = g_L b_R^{-1} = b_L g_R^{-1} \quad \text{with} \quad g_L, g_R \in G,\ b_L, b_R \in B,$$

which yield the ‘Iwasawa maps’ $\Xi_L, \Xi_R : G_{\mathbb{R}}^{\mathbb{C}} \to G$ and $\Lambda_L, \Lambda_R : G_{\mathbb{R}}^{\mathbb{C}} \to B$,

$$\Xi_L(K) := g_L, \quad \Xi_R(K) := g_R, \quad \Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R.$$

We have the diffeomorphic manifolds $M := G_{\mathbb{R}}^{\mathbb{C}}$, $\mathfrak{M} := G \times B$ and $\mathbb{M} := G \times \mathfrak{P}$. Shall use the diffeomorphisms $m_1 := (\Xi_R, \Lambda_R) : M \to \mathfrak{M}$, that is, $m_1(K) = (g_R, b_R)$, and $m_2 : \mathfrak{M} \to \mathbb{M}$, $m_2(g, b) := (g, bb^\dagger)$.

The map $\nu : B \ni b \mapsto bb^\dagger \in \mathfrak{P} = \exp(\mathrm{i}\mathcal{G}) \subset G_{\mathbb{R}}^{\mathbb{C}}$ is a G-equivariant diffeomorphism if G acts on $\mathfrak{P}$ by conjugations and on B by ‘dressing’: $\mathrm{Dress}_\eta(b) := \Lambda_L(\eta b)$, $\forall \eta \in G$, $b \in B$.

The group manifold $M = G_{\mathbb{R}}^{\mathbb{C}}$ carries the following two Poisson brackets:

$$\{\Phi_1, \Phi_2\}_\pm := \langle \nabla\Phi_1, \rho\nabla\Phi_2\rangle_{\mathbb{I}} \pm \langle \nabla'\Phi_1, \rho\nabla'\Phi_2\rangle_{\mathbb{I}}, \quad \forall \Phi_1, \Phi_2 \in C^\infty(M).$$

Here, $\rho := \frac{1}{2}(\pi_{\mathcal{G}} - \pi_{\mathcal{B}})$ with $\pi_{\mathcal{G}}$ and $\pi_{\mathcal{B}}$ denoting the projections from $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ onto $\mathcal{G}$ and $\mathcal{B}$, which correspond to the direct sum $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}} = \mathcal{G} + \mathcal{B}$. For any real function $\Phi \in C^\infty(M)$, the $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$-valued 'left- and right-derivatives' are defined by

$$\langle X, \nabla\Phi(K)\rangle_{\mathbb{I}} + \langle X', \nabla'\Phi(K)\rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} \Phi(e^{tX} K e^{tX'}), \quad \forall K \in M,\ X, X' \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}.$$

The minus bracket makes M into a Poisson–Lie group, of which G and B are Poisson–Lie subgroups, i.e., (embedded) Lie subgroups and Poisson submanifolds. Their inherited Poisson brackets take the form

$$\{\chi_1, \chi_2\}_G(g) = -\langle D'\chi_1(g), g^{-1}(D\chi_2(g))g\rangle_{\mathbb{I}},$$

$$\{\varphi_1, \varphi_2\}_B(b) = \langle D'\varphi_1(b), b^{-1}(D\varphi_2(b))b\rangle_{\mathbb{I}}.$$

The derivatives are $\mathcal{B}$-valued for $\chi_i \in C^\infty(G)$ and $\mathcal{G}$-valued for $\varphi_i \in C^\infty(B)$. The Poisson manifolds $(M, \{-,-\}_-)$ and $(M, \{-,-\}_+)$ are known, respectively, as the Drinfeld double and the Heisenberg double associated with the standard Poisson structures of G and B. The Poisson bracket $\{-,-\}_+$ is non-degenerate, its symplectic form reads

$$\Omega_+ = \frac{1}{2}\left\langle db_L b_L^{-1} \overset{\wedge}{,} dg_L g_L^{-1}\right\rangle_{\mathbb{I}} + \frac{1}{2}\left\langle db_R b_R^{-1} \overset{\wedge}{,} dg_R g_R^{-1}\right\rangle_{\mathbb{I}}.$$

The maps

$$(\Lambda_L, \Lambda_R) : M \to B \times B \quad \text{and} \quad (\Xi_L, \Xi_R) : M \to G \times G$$

are Poisson maps with respect to $(M, \{-,-\}_+)$ and the direct product Poisson structures on the targets obtained from $(B, \{-,-\}_B)$ and from $(G, \{-,-\}_G)$, respectively. (References: Semenov-Tian-Shansky [1985] and Alekseev–Malkin [1994]).

'Master system' on $\mathbb{M}$. For any real function $\phi \in C^\infty(\mathfrak{P})$, define its $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$-*valued derivative* $\mathcal{D}\phi$ as follows:

$$\langle X, \mathcal{D}\phi(L)\rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} \phi(e^{tX} L e^{tX^\dagger}),\ \langle Y, \mathcal{D}\phi(L)\rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} \phi(e^{tY} L e^{-tY}),\ \forall X \in \mathcal{B},\ Y \in \mathcal{G}.$$

Using the diffeomorphism $m := m_2 \circ m_1 : M \to \mathbb{M} = G \times \mathfrak{P}$, we transfer the Heisenberg double Poisson bracket to $\mathbb{M} = G \times \mathfrak{P}$. This gives

$$\{\mathcal{F}, \mathcal{H}\}_{\mathbb{M}}(g, L) = \langle \mathcal{D}_2\mathcal{F}, (\mathcal{D}_2\mathcal{H})_{\mathcal{G}}\rangle_{\mathbb{I}} - \left\langle g\mathcal{D}_1'\mathcal{F}g^{-1}, \mathcal{D}_1\mathcal{H}\right\rangle_{\mathbb{I}} + \langle \mathcal{D}_1\mathcal{F}, \mathcal{D}_2\mathcal{H}\rangle_{\mathbb{I}} - \langle \mathcal{D}_1\mathcal{H}, \mathcal{D}_2\mathcal{F}\rangle_{\mathbb{I}},$$

where the derivatives of $\mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{M})$ are evaluated at $(g, L) \in \mathbb{M}$; and $\mathcal{D}_1\mathcal{F} \in \mathcal{B}$.

Define the map $\Psi : \mathbb{M} \to \mathfrak{P} \times \mathfrak{P}$ by $\Psi(g, L) := (g^{-1}Lg, L)$, which in terms of the model M reads $(\nu \circ (\Lambda_L)^{-1}, \nu \circ \Lambda_R)$; remember $\nu(b) = bb^\dagger$.

Proposition 1. *The two subrings of $C^\infty(\mathbb{M})$ defined by*

$$\mathfrak{H} := \pi_2^*\left(C^\infty(\mathfrak{P})^G\right) \quad \textit{and} \quad \mathfrak{F} := \Psi^*\left(C^\infty(\mathfrak{P} \times \mathfrak{P})\right)$$

engender a degenerate integrable system on the symplectic manifold $(\mathbb{M}, \{-,-\}_{\mathbb{M}})$. The rank of this integrable system is equal to the rank $r = \dim(\mathcal{G}_0)$ of Lie algebra $\mathcal{G}$.

Proof. One calculates that any Hamiltonian $\mathcal{H} = \pi_2^*(\phi)$ with a function $\phi \in C^\infty(\mathfrak{P})^G$ has the integral curves

$$(g(t), L(t)) = (\exp\left(t\mathcal{D}\phi(L(0))\right) g(0), L(0)).$$

Since the derivative $\mathcal{D}\phi : \mathfrak{P} \to \mathcal{G}$ is G-equivariant, **Ψ is constant along these curves, and it is a Poisson map** for the ν-transferred Poisson bracket on $\mathfrak{P}_- \times \mathfrak{P}$. One can verify that the derivative $D\Psi$ has constant rank, equal to $\dim(\mathbb{M}) - r$, at every point of $G \times \mathfrak{P}^{\text{reg}}$. This implies that $\mathfrak{F}$ has functional dimension $\dim(\mathbb{M}) - r$. It is obvious that $\mathfrak{H} \subset \mathfrak{F}$, and its functional dimension is r, which completes the proof.

Reduction. Define the 'conjugation action' $A : G \times \mathbb{M}$ by $A_\eta(g, L) := (\eta g \eta^{-1}, \eta L \eta^{-1})$. All $\mathcal{H} \in \mathfrak{H}$ and their Hamiltonian vector fields are G-invariant, and the invariant functions, $C^\infty(\mathbb{M})^G$, form a Poisson subalgebra. Therefore, we may take the Poisson quotient

$$\mathbb{M}^{\mathrm{red}} := \mathbb{M}/G, \quad C^\infty(\mathbb{M}^{\mathrm{red}}) := C^\infty(\mathbb{M})^G.$$

$$\text{We have} \qquad \mathfrak{H} \subset \mathfrak{F}^G := \Psi^*(C^\infty(\mathfrak{P}_- \times \mathfrak{P})^G) \subset C^\infty(\mathbb{M})^G.$$

For $\mathbb{M}^{\mathrm{red}}$ is a not a smooth manifold, we restrict to its dense open subset $\mathbb{M}^{\mathrm{red}}_* = \mathbb{M}_*/G$, where $\mathbb{M}_* \subset \mathbb{M}$ is the submanifold of principal orbit type:

$\mathbb{M}_* := \{(g, L) \in \mathbb{M} \mid G_{(g,L)} = Z(G)\}$. Note: $\mathbb{M}_*$ is stable w.r.t. the flows of $C^\infty(\mathbb{M})^G$.

The' space of constants of motion' $\mathfrak{C} := \Psi(\mathbb{M}) \subset \mathfrak{P} \times \mathfrak{P}$ is also not a smooth manifold, but $\mathfrak{C}_{\mathrm{reg}} := \{\tilde{L}, L) \in \mathfrak{C} \mid L \in \mathfrak{P}^{\mathrm{reg}}$ is a smooth, embedded submanifold of $\mathfrak{P}^{\mathrm{reg}} \times \mathfrak{P}^{\mathrm{reg}}$. Here, $\mathfrak{P}^{\mathrm{reg}}$ consists of the points of $\mathfrak{P}$ whose isotropy group in G is a maximal torus.

A key technical point is to consider

$$\mathfrak{C}_* := \{(\tilde{L}, L) \in \mathfrak{C}_{\mathrm{reg}} \mid G_{(\tilde{L},L)} = Z(G)\} \quad \text{and} \quad \mathbb{M}_{**} := \Psi^{-1}(\mathfrak{C}_*).$$

The restriction of Ψ yields the G-equivariant submersion $\psi : \mathbb{M}_{**} \to \mathfrak{C}_*$, and we get the diagram of **smooth** Poisson submersions (where $\mathbb{M}^{\mathrm{red}}_{**} = \mathbb{M}_{**}/G$ and $\mathfrak{C}^{\mathrm{red}}_* = \mathfrak{C}_*/G$):

$$\begin{array}{ccc} \mathbb{M}_{**} & \xrightarrow{\quad \psi \quad} & \mathfrak{C}_* \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathbb{M}^{\mathrm{red}}_{**} & \xrightarrow{\quad \psi_{\mathrm{red}} \quad} & \mathfrak{C}^{\mathrm{red}}_* \end{array}$$

The rings $\mathfrak{H}$ and $\mathfrak{F}^G$ yield the subrings $\mathfrak{H}_{\text{red}}$ and $\mathfrak{F}_{\text{red}}$ of $C^\infty(\mathbb{M}^{\text{red}})$, and we denote their restrictions on $\mathbb{M}_*^{\text{red}}$ and $\mathbb{M}_{**}^{\text{red}}$ by $\mathfrak{H}_{\text{red}}^*$, $\mathfrak{H}_{\text{red}}^{**}$ and $\mathfrak{F}_{\text{red}}^*$, $\mathfrak{F}_{\text{red}}^{**}$, respectively. Moreover, we define the restricted reduced Poisson manifold by

$$(C^\infty(\mathbb{M}_{**}^{\text{red}}), \{-,-\}_{**}^{\text{red}}) \simeq (C^\infty(\mathbb{M}_{**})^G, \{-,-\}_{\mathbb{M}_{**}}).$$

Theorem 2. *Suppose that $r := \dim(\mathcal{G}_0) \neq 1$. Then, the 'restricted reduced system' $(C^\infty(\mathbb{M}_{**}^{\text{red}}), \{-,-\}_{**}^{\text{red}}), \mathfrak{H}_{\text{red}}^{**})$ is a degenerate integrable system of rank r with constants of motion provided by the ring of functions*

$$\mathfrak{F}_{\text{red}}^\sharp := \psi_{\text{red}}^* \left(C^\infty(\mathfrak{C}_*^{\text{red}}) \right).$$

That is, the quadruple $(\mathbb{M}_{**}^{\text{red}}, \{-,-\}_{**}^{\text{red}}, \mathfrak{H}_{\text{red}}^{**}, \mathfrak{F}_{\text{red}}^\sharp)$ satisfies Definition I, with the co-dimension of the generic symplectic leaves being equal to r. The reduced Hamiltonian vector fields associated with $\mathfrak{H}_{\text{red}}^{**}$ span an r-dimensional subspace of the tangent space at every point of $\mathbb{M}_{**}^{\text{red}}$, and the differentials of the elements of $\mathfrak{F}_{\text{red}}^\sharp$ span a co-dimension r subspace of the cotangent space.

The symplectic leaves in $\mathbb{M}_*^{\text{red}}$ as well as in $\mathbb{M}_{**}^{\text{red}}$ are (the connected components of) the joint level surfaces of the Casimir functions, which are obtained from

$$\mathit{\Lambda}^*(C^\infty(B)^G) \quad \text{with the Poisson–Lie moment map} \quad \mathit{\Lambda} : \mathbb{M} \to B.$$

The map $\mathit{\Lambda}$ is defined by transferring to $\mathbb{M}$ the moment map $\Lambda := \Lambda_L \Lambda_R : G_{\mathbb{R}}^{\mathbb{C}} \to B$. The conjugation action of G is orbit-equivalent to the Poisson–Lie action generated by the moment map.

Corollary 3. *The restriction of the system $(\mathbb{M}_{**}^{\text{red}}, \{-,-\}_{**}^{\text{red}}, \mathfrak{H}_{\text{red}}^{**}, \mathfrak{F}_{\text{red}}^\sharp)$ of Theorem 2 to any symplectic leaf of $\mathbb{M}_{**}^{\text{red}}$ of co-dimension r is a degenerate integrable system of rank r.*

Remark: The $r = 1$ case arises for $G = \text{SU}(2)$, and in this case we obtain 'only' Liouville integrability.

The integrability statement can be extended to $\mathbb{M}_*^{\rm red}$ by using that at each $y \in \mathbb{M}_{**}^{\rm red}$ the differentials of the elements of $\mathfrak{F}_{\rm red}^{**} \subset \mathfrak{F}_{\rm red}^{\sharp}$ span the same subspace of $T_y\mathbb{M}_{**}^{\rm red}$ as do the differentials of the elements of $\mathfrak{F}_{\rm red}^{\sharp}$. This can be shown utilizing the fact that for any smooth action of a compact Lie group on a connected manifold the dimension of the differentials of the smooth invariant functions at a point of principal orbit type is equal to the co-dimension of the principal orbits. (We apply this to $C^\infty(\mathfrak{P}_- \times \mathfrak{P})^G$ and use pull-back.) The point is that the elements of $\mathfrak{F}_{\rm red}$ belong to $C^\infty(\mathbb{M}^{\rm red})$ and their restrictions give smooth function on $\mathbb{M}_*^{\rm red}$.

Theorem 4. *Suppose that $r = \mathrm{rank}(G) > 1$ and consider the restriction of the master system of free motion on the dense, open submanifold $\mathbb{M}_* \subset \mathbb{M}$ of principal orbit type with respect to the G-action. Then, this system descends to the degenerate integrable system $(\mathbb{M}_*^{\rm red}, \{-,-\}_*^{\rm red}, \mathfrak{H}_{\rm red}^*, \mathfrak{F}_{\rm red}^*)$ on the Poisson manifold $\mathbb{M}_*^{\rm red} = \mathbb{M}_*/G$, where the Poisson subalgebras $\mathfrak{H}_{\rm red}^*$ and $\mathfrak{F}_{\rm red}^*$ of $C^\infty(\mathbb{M}_*^{\rm red}) = C^\infty(\mathbb{M}_*)^G$ arise from the restrictions of $\mathfrak{H}$ and $\mathfrak{F}_{\rm red} \simeq \Psi^*(C^\infty(\mathfrak{P}_- \times \mathfrak{P})^G)$ on $\mathbb{M}_* \subset \mathbb{M}$, respectively.*

Dynamical r-matrix formula for reduced Poisson brackets. Restrict to the dense, open, G-invariant submanifold $\pi_1^{-1}(G^{\mathrm{reg}}) = G^{\mathrm{reg}} \times \mathfrak{P} \subset \mathbb{M}$. Every G-orbit in this submanifold intersects $\mathbb{M}_0 := \{(Q,L) \in \mathbb{M} \mid Q \in G_0^{\mathrm{reg}}\}$. The intersection happens in orbits of the normalizer $\mathfrak{N} := N_G(G_0)$, and we obtain the identifications

$$(G^{\mathrm{reg}} \times \mathfrak{P})/G \simeq \mathbb{M}_0/\mathfrak{N} \quad \text{and} \quad C^\infty(G^{\mathrm{reg}} \times \mathfrak{P})^G \Longleftrightarrow C^\infty(\mathbb{M}_0)^{\mathfrak{N}}.$$

Let $\bar{\mathcal{F}}, \bar{\mathcal{H}} \in C^\infty(\mathbb{M}_0)^{\mathfrak{N}}$ be the restrictions of $\mathcal{F}, \mathcal{H} \in C^\infty(G^{\mathrm{reg}} \times \mathfrak{P})^G$. Then, we define their 'reduced Poisson bracket' by

$$\{\bar{\mathcal{F}}, \bar{\mathcal{H}}\}^{\mathrm{red}}_{\mathbb{M}_0}(Q,L) := \{\mathcal{F}, \mathcal{H}\}_{\mathbb{M}}(Q,L), \qquad \forall (Q,L) \in \mathbb{M}_0.$$

Its explicit form contains the dynamical r-matrix $\mathcal{R}(Q) \in \mathrm{End}(\mathcal{G}^{\mathbb{C}}_{\mathbb{R}})$:

$$\mathcal{R}(Q)(X) := \frac{1}{2}(\mathrm{Ad}_Q + \mathrm{id}) \circ (\mathrm{Ad}_Q - \mathrm{id})^{-1}_{|\mathcal{G}^{\mathbb{C}}_\perp}(X_\perp), \qquad \forall Q \in G_0^{\mathrm{reg}},\ X = (X_0 + X_\perp) \in \mathcal{G}^{\mathbb{C}}_{\mathbb{R}},$$

where $X_0 \in \mathcal{G}_0^{\mathbb{C}}$ and $X_\perp \in \mathcal{G}_\perp^{\mathbb{C}}$, in correspondence with $\mathcal{G}^{\mathbb{C}} = \mathcal{G}_0^{\mathbb{C}} + \mathcal{G}_\perp^{\mathbb{C}}$.

Theorem 5. *For $\bar{\mathcal{F}}, \bar{\mathcal{H}} \in C^\infty(\mathbb{M}_0)^{\mathfrak{N}}$, the definition implies the formula*

$$\{\bar{\mathcal{F}}, \bar{\mathcal{H}}\}^{\mathrm{red}}_{\mathbb{M}_0}(Q,L) = \langle \mathcal{D}_1\bar{\mathcal{F}}, \mathcal{D}_2\bar{\mathcal{H}}\rangle_{\mathbb{I}} - \langle \mathcal{D}_1\bar{\mathcal{H}}, \mathcal{D}_2\bar{\mathcal{F}}\rangle_{\mathbb{I}} + \langle \mathcal{R}(Q)\mathcal{D}_2\bar{\mathcal{H}}, \mathcal{D}_2\bar{\mathcal{F}}\rangle_{\mathbb{I}},$$

where the derivatives $\mathcal{D}_1\bar{\mathcal{F}} \in \mathcal{B}_0$ and $\mathcal{D}_2\bar{\mathcal{F}} \in \mathcal{G}^{\mathbb{C}}_{\mathbb{R}}$ are taken at (Q,L). The Hamiltonian $\bar{\mathcal{H}}(Q,L) = \phi(L)$ with $\phi \in C^\infty(\mathfrak{P})^G$ induces the evolution equations

$$\dot{Q} = (\mathcal{D}\phi(L))_0 Q, \quad \dot{L} = [\mathcal{R}(Q)\mathcal{D}\phi(L), L] \qquad \textit{(up to residual gauge transformations).}$$

The formula defines a Poisson algebra structure on $C^\infty(\mathbb{M}_0)^{G_0}$ as well. For some purposes, it is advantageous to use, instead of $\mathbb{M}_0 = G_0^{\mathrm{reg}} \times \mathfrak{P}$, the equivalent model $\mathfrak{M}_0 := G_0^{\mathrm{reg}} \times B$. Then, the reduced Poisson bracket becomes

$$\{\bar{f}, \bar{h}\}^{\mathrm{red}}_{\mathfrak{M}_0}(Q,b) = \langle D_1\bar{f}, D_2\bar{h}\rangle_{\mathbb{I}} - \langle D_1\bar{h}, D_2\bar{f}\rangle_{\mathbb{I}}, + \langle \mathcal{R}(Q)(bD_2'\bar{h}b^{-1}), bD_2'\bar{f}b^{-1}\rangle_{\mathbb{I}}.$$

Here, the derivatives are evaluated at (Q,b), with $D_1\bar{f} \in \mathcal{B}_0$ and $D_2\bar{f}, D_2'\bar{f} \in \mathcal{G}$.

Canonically conjugate pairs and 'spin' variables. Let B_0 and B_+ be the subgroups of B associated with the subalgebras in $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_+$. Any $b \in B$ is uniquely decomposed as $b = e^p b_+$ with $p \in \mathcal{B}_0$, $b_+ \in B_+$. Then, we introduce new variables by means of the map

$$\zeta : \mathfrak{M}_0 = G_0^{\text{reg}} \times B \to G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+$$

$$\zeta : (Q, e^p b_+) \mapsto (Q, p, \lambda) \quad \text{with} \quad \lambda := b_+^{-1} Q^{-1} b_+ Q.$$

The map ζ is a diffeomorphism. It is equivariant with respect to the G_0-actions for which $\eta_0 \in G_0$ sends (Q, b) to $(Q, \eta_0 b \eta_0^{-1})$ and (Q, p, λ) to $(Q, p, \eta_0 \lambda \eta_0^{-1})$. Consequently, ζ induces an isomorphism: $C^\infty(\mathfrak{M}_0)^{G_0} \Longleftrightarrow C^\infty(G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$.

Any two functions $F, H \in C^\infty(G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$ are related to unique $\bar{f}, \bar{h} \in C^\infty(\mathfrak{M}_0)^{G_0}$ by $F \circ \zeta = \bar{f}$, $H \circ \zeta = \bar{h}$. Thus, we can define $\{F, H\}_0^{\text{red}} \in C^\infty(G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$ by

$$\{F, H\}_0^{\text{red}} \circ \zeta := \{\bar{f}, \bar{h}\}_{\mathfrak{M}_0}^{\text{red}}.$$

Theorem 6. *In terms of the new variables introduced via the map ζ, the reduced Poisson bracket acquires the following 'decoupled form':*

$$\{F, H\}_0^{\text{red}}(Q, p, \lambda) = \langle D_Q F, d_p H \rangle_{\mathbb{I}} - \langle D_Q H, d_p F \rangle_{\mathbb{I}} + \langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H \rangle_{\mathbb{I}},$$

where the derivatives of $F, H \in C^\infty(G_0^{\text{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}$ are taken at (Q, p, λ).

Using the identification $(\mathcal{B}_+)^* \simeq \mathcal{G}_\perp$, the derivatives $D_\lambda F, D'_\lambda F \in \mathcal{G}_\perp$ are defined by

$$\langle X_+, D_\lambda F(Q, p, \lambda) \rangle_{\mathbb{I}} + \langle X'_+, D'_\lambda F(Q, p, \lambda) \rangle_{\mathbb{I}} = \left.\frac{d}{dt}\right|_{t=0} F(Q, p, e^{tX_+} \lambda e^{tX'_+}), \ \forall X_+, X'_+ \in \mathcal{B}_+.$$

Comparison with the reduction of T^*G. The 'linear analogue' of the Poisson algebra $(C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)^{G_0}, \{-,-\}_0^{\mathrm{red}})$,

$$\{F,H\}_0^{\mathrm{red}}(Q,p,\lambda) = \langle D_Q F, d_p H\rangle_{\mathbb{I}} - \langle D_Q H, d_p F\rangle_{\mathbb{I}} + \langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H\rangle_{\mathbb{I}},$$

is given by $(C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+)^{G_0}, \{-,-\}_{\mathrm{lin}})$ with

$$\{f,h\}_{\mathrm{lin}}(Q,p,X) := \langle D_Q f, d_p h\rangle_{\mathbb{I}} - \langle D_Q h, d_p f\rangle_{\mathbb{I}} + \langle X, [d_X f, d_X h]\rangle_{\mathbb{I}},$$

where the derivatives are taken at (Q,p,X), and $d_X f \in \mathcal{G}_\perp \simeq (\mathcal{B}_+)^*$ denotes the differential of f with respect to its third variable. An interpretation of these brackets comes by observing that $B \simeq G^*$ and $\mathcal{B} \simeq \mathcal{G}^*$, and the reductions of $(B, \{-,-\}_B)$ and $(\mathcal{G}^*, \{-,-\}_{\mathcal{G}^*})$ with respect to the Hamiltonian actions of G_0, at the zero value of the $\mathcal{G}_0^*$-valued moment map, give precisely the third term of the respective Poisson brackets, i.e., they represent $G^*//G_0$ and $\mathcal{G}^*//G_0$, respectively. [Beware, in Lecture 1 we used the alternative model $\mathcal{G}^* \simeq \mathcal{G}$. Thus, $\xi \in \mathcal{G}_\perp$ used before is now replaced by $X \in \mathcal{B}_+$.]

The Poisson algebra $(C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+)^{G_0}, \{-,-\}_{\mathrm{lin}})$ arises from the Poisson reduction of the cotangent bundle T^*G by the obvious conjugation action, whereby the kinetic energy of the bi-invariant Riemannian metric of G reduces to the **spin Sutherland Hamiltonian**:

$$H_{\mathrm{spin-Suth}}(e^{\mathrm{i}q}, p, X) = \frac{1}{2}\langle p,p\rangle + \frac{1}{8}\sum_{\alpha\in\mathfrak{R}^+} \frac{1}{|\alpha|^2}\frac{|X_\alpha|^2}{\sin^2(\alpha(q)/2)} \quad \text{with} \quad X = \sum_{\alpha\in\mathfrak{R}^+} X_\alpha E_\alpha \in \mathcal{B}_+.$$

Proposition 7. *For any real $\epsilon > 0$, let us define the G_0-equivariant diffeomorphism*

$$\mu_\epsilon : G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times \mathcal{B}_+ \to G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+, \quad \mu_\epsilon : (Q,p,X) \mapsto (Q, \epsilon p, \exp(\epsilon X)).$$

Then, $\{-,-\}_{\mathrm{lin}}$ is the 'scaling limit' of $\{-,-\}_0^{\mathrm{red}}$ according to the formula

$$\{f,h\}_{\mathrm{lin}} = \lim_{\epsilon\to 0} \epsilon\{f\circ\mu_\epsilon^{-1}, h\circ\mu_\epsilon^{-1}\}_0^{\mathrm{red}} \circ \mu_\epsilon.$$

Interpretation as spin RS model: Consider the new variable $\lambda = b_+^{-1} Q^{-1} b_+ Q$ using

$$\lambda = e^{\sigma}, \quad b_+ = e^{\beta}, \quad \sigma = \sum_{\alpha>0} \sigma_\alpha E_\alpha, \quad \beta = \sum_{\alpha>0} \beta_\alpha E_\alpha, \quad Q = e^{\mathrm{i}q}.$$

The Baker-Campbell-Hausdorff formula gives

$$\exp(-\beta + Q^{-1}\beta Q + \frac{1}{2}[Q^{-1}\beta Q, \beta] + \cdots) = \exp(\sigma).$$

As a consequence, β_α can be expressed in terms of σ and $e^{\mathrm{i}q}$:

$$\beta_\alpha = \frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)} - 1} + \sum_{k\geq 2} \sum_{\varphi_1,\dots,\varphi_k} f_{\varphi_1,\dots,\varphi_k}(e^{\mathrm{i}q}) \sigma_{\varphi_1} \dots \sigma_{\varphi_k},$$

where $\alpha = \varphi_1 + \cdots + \varphi_k$ and $f_{\varphi_1,\dots,\varphi_k}$ depends rationally on $e^{\mathrm{i}q}$. This gives a construction of the inverse of the map $\zeta : (Q, e^p b_+) \to (Q, p, \lambda)$.

Take any finite dimensional irreducible representation $\rho : G^{\mathbb{C}} \to \mathrm{SL}(V)$. Introduce an inner product on V so that the dagger, $K^\dagger = \Theta(K^{-1})$, becomes the usual adjoint. Then, the (normalized) character $\phi^\rho(L) = \mathrm{tr}_\rho(L) := c_\rho \mathrm{tr} \rho(L)$ gives an element of $C^\infty(\mathfrak{P})^G$. (Here, c_ρ is a constant, so that $\mathrm{tr}_\rho(XY) := c_\rho \mathrm{tr}(\rho(X)\rho(Y)) = \langle X, Y \rangle$, $\forall X, Y \in \mathcal{G}^{\mathbb{C}}$.)

Using the 'decoupled variables' (Q, p, σ), $H^\rho := \mathrm{tr}_\rho(e^p b_+ b_+^\dagger e^p)$ can be expanded as

$$H^\rho(e^{\mathrm{i}q}, p, \sigma) = c_\rho \mathrm{tr} \left(e^{2p} \left(1_\rho + \frac{1}{4} \sum_{\alpha>0} \frac{|\sigma_\alpha|^2 \rho(E_\alpha)\rho(E_{-\alpha})}{\sin^2(\alpha(q)/2)} + o_2(\sigma, \sigma^*) \right) \right).$$

I call this a spin Ruijsenaars–Schneider (RS) type Hamiltonian.

By expanding e^{2p},

$$H^\rho(e^{\mathrm{i}q}, p, \sigma) = c_\rho \mathrm{dim}_\rho + 2\mathrm{tr}_\rho(p^2) + \frac{1}{2}\sum_{\alpha>0} \frac{1}{|\alpha|^2} \frac{|\sigma_\alpha|^2}{\sin^2(\alpha(q)/2)} + \mathrm{o}_2(\sigma, \sigma^*, p).$$

Leading term of $\frac{1}{4}(H^\rho - c_\rho \mathrm{dim}_\rho)$ matches the Hamiltonian $H_{\mathrm{spin-Suth}}(e^{\mathrm{i}q}, p, X)$. In other words, with the 'scaling map' μ_ϵ, we have

$$H_{\mathrm{spin-Suth}} = \lim_{\epsilon\to 0} \frac{1}{4\epsilon^2}(H^\rho \circ \mu_\epsilon - c_\rho \mathrm{dim}_\rho).$$

The Poisson brackets of the functions of the 'spin variables' X and σ follow from

$$\{X^i, X^j\}_{\mathcal{G}^*}(X) = \langle [Y^i, Y^j], X\rangle_{\mathbb{I}}, \quad \{\sigma^i, \sigma^j\}_B(e^\sigma) = \langle [Y^i, Y^j], \sigma\rangle_{\mathbb{I}} + \mathrm{o}(\sigma),$$

where $X^i = \langle X, Y^i\rangle_{\mathbb{I}}$ for a basis $\{Y^i\}$ of $\mathcal{G}_\perp \subset \mathcal{G} = \mathcal{G}_0 + \mathcal{G}_\perp$, and similarly for σ. Proposition 7 is a consequence of the latter expansion.

The elements of $C^\infty(\mathfrak{P})^G$ yield G-invariant functions of 'Lax matrix' $L(e^{\mathrm{i}q}, p, \sigma) := e^p b_+ b_+^\dagger e^p$, where $b_+ = b_+(e^{\mathrm{i}q}, \sigma)$ expresses the inverse of our map ζ. In any representation,

$$L(e^{\mathrm{i}q}, p, \sigma) = 1 + 2p + \sum_{\alpha>0}\left(\frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)} - 1} E_\alpha + \frac{\sigma_\alpha^*}{e^{\mathrm{i}\alpha(q)} - 1} E_{-\alpha}\right) + \mathrm{o}(\sigma, \sigma^*, p).$$

This matches the standard, $\mathcal{G}$-valued, spin Sutherland Lax matrix.

In conclusion, our models are 'deformations' of the spin Sutherland models, which can be recovered in the 'scaling limit'.

Explicit formulas for $G^{\mathbb{C}} = \mathrm{SL}(n,\mathbb{C})$: Now parametrize $b_+ \in B$ by its matrix elements. We have $b = e^p b_+$, and can find b_+ from the relation

$$Q^{-1} b_+ Q = b_+ \lambda,$$

where $Q = \mathrm{diag}(Q_1, \ldots, Q_n) \in G_0^{\mathrm{reg}}$, $\lambda \in B_+$ is the constrained 'spin' variable and b_+ is an upper triangular matrix with unit diagonal.

Introducing $\mathcal{I}_{a,a+j} := \frac{1}{Q_{a+j}Q_a^{-1} - 1}$, we have $(b_+)_{a,a+1} = \mathcal{I}_{a,a+1}\lambda_{a,a+1}$, and, for $k = 2, \ldots, n-a$, the matrix element $(b_+)_{a,a+k}$ equals

$$\mathcal{I}_{a,a+k}\lambda_{a,a+k} + \sum_{\substack{m=2,\ldots,k \\ (i_1,\ldots,i_m)\in\mathbb{N}^m \\ i_1+\cdots+i_m=k}} \prod_{\alpha=1}^{m} \mathcal{I}_{a,a+i_1+\cdots+i_\alpha}\lambda_{a+i_1+\cdots+i_{\alpha-1},a+i_1+\cdots+i_\alpha}.$$

Then $H = \mathrm{tr}(bb^\dagger)$ gives

$$H(e^{\mathrm{i}q}, p, \lambda) = \sum_{a=1}^{n} e^{2p_a} + \frac{1}{4}\sum_{a=1}^{n-1} e^{2p_a} \sum_{k=1}^{n-a} \frac{|\lambda_{a,a+k}|^2}{\sin^2((q_{a+k}-q_a)/2)} + \mathrm{o}_2(\lambda, \lambda^\dagger).$$

Next, we explain that restricting λ to a minimal dressing orbit of $\mathrm{SU}(n)$ results in the standard (spinless) real, trigonometric Ruijsenaars–Schneider model.

Taking $G = \mathrm{SU}(n)$, let us go back to

$$\{F,H\}_0^{\mathrm{red}}(Q,p,\lambda) = \langle D_Q F, d_p H\rangle_{\mathbb{I}} - \langle D_Q H, d_p F\rangle_{\mathbb{I}} + \langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H\rangle_{\mathbb{I}},$$

and restrict λ to a minimal dressing orbit. This is the orbit $\mathcal{O}(y) \subset B(n)$ through

$$\Delta(y) := \exp\left(\mathrm{diag}((n-1)y/2, -y/2, \cdots, -y/2)\right), \quad \text{for some} \quad y \in \mathbb{R}^*.$$

It turns out that

$$\mathcal{O}(y) \cap B(n)_+ = \{T\nu(y)T^{-1} \mid T \in G_0\},$$

with the matrix $\nu(y) \in B(n)_+$ given by $\nu(y)_{jk} = (1-e^{-y})\exp((k-j)y/2)$, $\forall j < k$. Therefore the G_0-reduced orbit now consist of a single point, and the reduced Poisson (symplectic) structure is encoded by

$$\{F,H\}_0^{\mathrm{red}}(Q,p) = \langle D_Q F, d_p H\rangle_{\mathbb{I}} - \langle D_Q H, d_p F\rangle_{\mathbb{I}}.$$

For fixed $\lambda = \nu(y)$ and Q, the equation $b_+^{-1}Q^{-1}b_+Q = \nu(y)$ determines b_+. We find

$$(b_+)_{kl} = Q_k \bar{Q}_l \prod_{m=1}^{l-k} \frac{e^{\frac{y}{2}}\bar{Q}_k - e^{-\frac{y}{2}}\bar{Q}_{k+m-1}}{\bar{Q}_k - \bar{Q}_{k+m}}, \quad 1 \le k < l \le n, \quad \bar{Q}_k = Q_k^{-1} = e^{-\mathrm{i}q_k}.$$

Then, after the canonical transformation $(q,p) \to (q,\theta)$ with

$$\theta_k = p_k - \frac{1}{4}\sum_{m<k} \ln\left[1 + \frac{\sinh^2(y/2)}{\sin^2((q_k-q_m)/2)}\right] + \frac{1}{4}\sum_{m>k} \ln\left[1 + \frac{\sinh^2(y/2)}{\sin^2((q_k-q_m)/2)}\right],$$

we obtain the trigonometric Ruijsenaars–Schneider Hamiltonian from $b = e^p b_+$:

$$H_{\mathrm{RS}}(q,\theta) := \sum_{k=1}^{n} \cosh(2\theta_k) \prod_{m\neq k} \left[1 + \frac{\sinh^2(y/2)}{\sin^2((q_k-q_m)/2)}\right]^{\frac{1}{2}} = \frac{1}{2}\mathrm{tr}(bb^\dagger) + (bb^\dagger)^{-1}).$$

The symplectic leaf is $T^*G_0^{\mathrm{reg}}/S_n$ and (q,θ) parametrizes $T^*G_0^{\mathrm{reg}}$, which motivated the transformation.

The dual system in a nutshell

We have the following 3 models of the Heisenberg double

$$G_{\mathbb{R}}^{\mathbb{C}} \simeq G \times B \simeq G \times \mathfrak{P}.$$

To study the 'dual master system', the first model, $M = G_{\mathbb{R}}^{\mathbb{C}}$, is convenient.

Recall that $K \in G_{\mathbb{R}}^{\mathbb{C}}$ admits the Iwasawa decompositions

$$K = g_L b_R^{-1} = b_L g_R^{-1} \quad \text{with} \quad g_L, g_R \in G,\, b_L, b_R \in B,$$

which yield the 'Iwasawa maps' $\Xi_L, \Xi_R : G_{\mathbb{R}}^{\mathbb{C}} \to G$ and $\Lambda_L, \Lambda_R : G_{\mathbb{R}}^{\mathbb{C}} \to B$,

$$\Xi_L(K) := g_L, \quad \Xi_R(K) := g_R, \quad \Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R.$$

The Abelian Poisson algebra of the 'dual system' is $\tilde{\mathfrak{H}} := \Xi_R^*(C^\infty(G)^G)$. To describe the integral curve of $\Xi_R^*(\chi) \in \tilde{\mathfrak{H}}$ through $K(0) \in G_{\mathbb{R}}^{\mathbb{C}}$, we need the decomposition

$$\exp(\mathrm{i}t\nabla\chi(g_R(0))) = \beta(t)^{-1}\gamma(t) \quad \text{with} \quad \beta(t) \in B,\ \gamma(t) = G.$$

For the class function $\chi \in (C^\infty(G)^G)$, we use the $\mathcal{G}$-valued derivative $\nabla\chi$ defined by $\langle X, \nabla\chi(g)\rangle := \frac{d}{dt}\big|_{t=0} \chi(e^{tX}g), \quad \forall g \in G,\ X \in \mathcal{G}$. Then, the integral curve is

$$K(t) = K(0)\beta(t)^{-1} \longleftrightarrow b_R(t) = \beta(t)b_R(0),\ b_L(t) = b_L(0)\beta(t)^{-1},\ g_L(t) = g_L(0),$$

and $g_R(t) = \gamma(t)g_R(0)\gamma(t)^{-1}$. Since $L(t) = b_R(t)b_R(t)^\dagger = \beta(t)L(0)\beta(t)^\dagger$, we also have the integral curve in terms of the model $G \times \mathfrak{P}$.

In this case, we have the **map of constants of motion**

$$\tilde{\Psi} : G_{\mathbb{R}}^{\mathbb{C}} \to G_{\mathbb{R}}^{\mathbb{C}} \quad \text{defined by} \quad \tilde{\Psi}(K) := b_L b_R g_L^{-1} \equiv b_L g_R b_L^{-1}.$$

This is equivariant with respect to the conjugation action of G on the target space $G_{\mathbb{R}}^{\mathbb{C}}$ and the action of G on the Heisenberg double that is induced by the Poisson-Lie moment map $\Lambda = \Lambda_L \Lambda_R$. The $\tilde{\Psi}$-pullback of the ring of invariants

$$C^\infty(G_{\mathbb{R}}^{\mathbb{C}})^G := \{F \in C^\infty(G_{\mathbb{R}}^{\mathbb{C}}) \mid F(\eta K \eta^{-1}) \ \forall \eta \in G,\ K \in G_{\mathbb{R}}^{\mathbb{C}}\}$$

yield constants of motion that descend to the reduced phase space. These guarantee the degenerate integrability of the dual master system and its Poisson reduction.

Let me finish by mentioning the example of dual Ruijsenaars–Schneider system, given by the 'main Hamiltonian'

$$\tilde{H}_{\mathrm{RS}} := \sum_{k=1}^{n} \cos(2\hat{\theta}_k) \prod_{m \neq k} \left[1 - \frac{\sinh^2(y/2)}{\sinh^2((\hat{q}_k - \hat{q}_m)/2)}\right]^{\frac{1}{2}}.$$

To interpret this, we consider $G = SU(n)$ and pick the same symplectic leaf as before, which belongs to the specific moment map value $\nu(y)$.

In fact, [LF-Klimcik 2011], $\tilde{H}_{\mathrm{RS}}$ descends from the class function $\chi(g) := \frac{1}{2}\Re(\mathrm{tr}(g))$. The 'dual position variables $\hat{q}_k$ arise from the eigenvalues of $L = b_R b_R^\dagger$. This formula of the reduced Hamiltonian is valid on a dense open subset. It was shown by Ruijsenaars in 1995 that $\tilde{H}_{\mathrm{RS}}$ is Liouville integrable on its complete(d) phase space, and this result received a natural interpretation in the reduction approach.

This exemplifies the so-called Ruijsenaars duality (or action-position duality) between two integrable many-body systems.

Conclusion and open questions

1. I constructed 'Poisson–Lie deformations' of trigonometric spin Sutherland models.

2. I proved their degenerate integrability after restriction on the honest Poisson manifold $\mathbb{M}_*^{\rm red} \subset \mathbb{M}^{\rm red}$ as well as on the maximal symplectic leaves of the open dense subset $\mathbb{M}_{**}^{\rm red} \subset \mathbb{M}_*^{\rm red}$.

3. For lack of time, I did not present it, but recently I also proved integrability on arbitrary symplectic leaves of $\mathbb{M}_{**}^{\rm red}$ (by a different method). This new method will be reported in my talk at the workshop.

4. Quantization by quantum Hamiltonian reduction?

5. An old open question: Can one derive the spinless (real, repulsive) hyperbolic RS model by Hamiltonian reduction of a **real** master integrable system?

In this Appendix, we sketch a generalization of the trigonometric Gibbons–Hermsen model. For this, recall the GH model is obtained by Hamiltonian reduction from

$$T^*U(n) \times \mathbb{C}^{n\times d}.$$

The second factor encodes nd ($d \geq 2$) copies of the symplectic vector space $\mathbb{R}^2$. Denote the general element of $\mathbb{C}^{n\times d}$ as the matrix S_{aj}, and let (g, J) stand for the general element of the cotangent bundle, trivialized by right-translations. Then the following formula gives a Poisson map into $\mathfrak{u}(n) \simeq \mathfrak{u}(n)^*$,

$$\Phi(g, J, S) = J - g^{-1}Jg + \mathrm{i}SS^\dagger.$$

This is the moment map for the Hamiltonian action of $U(n)$ given by

$$A_\eta : (g, J, S) \mapsto (\eta g \eta^{-1}, \eta J \eta^{-1}, \eta S), \quad \forall \eta \in U(n).$$

Now, reduce by imposing the moment map constraint $\Phi(g, J, S) = \mathrm{i}c\mathbf{1}_n$, with $c > 0$. On a dense open subset, one can employ the partial gauge fixing where $g = \exp(\mathrm{i}q) \in \mathbb{T}^n_{\mathrm{reg}}$ with the maximal torus $\mathbb{T}^n < U(n)$. Then, one gets

$$J_{ab} = \mathrm{i}p_a\delta_{ab} - \mathrm{i}(1 - \delta_{ab})\frac{S_aS_b^\dagger}{1 - \exp(\mathrm{i}(q_b - q_a))}, \quad \text{with arbitrary} \quad p_a \in \mathbb{R}.$$

In this gauge, the 'free' Hamiltonian gives $H = -\frac{1}{2}\mathrm{tr}(J^2) = \frac{1}{2}\sum_{a=1}^n p_a^2 + \frac{1}{8}\sum_{a\neq b}\frac{|S_aS_b^\dagger|^2}{\sin^2\frac{q_a-q_b}{2}}$, and the moment map constraint implies $S_aS_a^\dagger = c$. The residual gauge transformations are given by the torus $\mathbb{T}^n$ and by the permutation group S_n, and the pertinent open dense subset of the full reduced phase space can be identified as

$$\left(T^*\mathbb{T}^n_{\mathrm{reg}} \times (\mathbb{CP}^{d-1} \times \cdots \times \mathbb{CP}^{d-1})\right)/S_n,$$

with n-copies of the complex projective space. (If $d = 1$, then one gets the spinless Sutherland model.)

For generalization, take the unreduced phase space $\mathcal{M} := GL(n,\mathbb{C}) \times \mathbb{C}^{n\times d}$, where the real manifold $GL(n,\mathbb{C}) \simeq U(n) \times \mathfrak{P}(n)$ is the Heisenberg double of the Poisson–Lie group $U(n)$ and the d columns of $\mathbb{C}^{n\times d}$ carry a $U(n)$ covariant Poisson structure,

$$\{w_k, w_l\} = \mathrm{i}\,\mathrm{sgn}(k-l) w_k w_l, \qquad \forall 1 \le k, l \le n,$$
$$\{w_k, \overline{w}_l\} = \mathrm{i}\,\delta_{kl}(2 + |w|^2) + \mathrm{i} w_k \overline{w}_l + \mathrm{i}\,\delta_{kl} \sum_{r=1}^{n} \mathrm{sgn}(r-k)|w_r|^2,$$

which is due to Zakrzewski (1996), and is actually symplectic. Consider the following Iwasawa decompositions of $K \in GL(n,\mathbb{C})$ and the factorization of $(\mathbf{1}_n + ww^\dagger) \in \mathfrak{P}(n)$:

$$K = g_L b_R^{-1} = b_L g_R^{-1}, \quad \mathbf{1}_n + ww^\dagger = \mathbf{b}(w)\mathbf{b}(w)^\dagger$$

where $g_L, g_R \in U(n)$ and $b_L, b_R, \mathbf{b}(w) \in B(n)$: the upper-triangular subgroup of $GL(n,\mathbb{C})$ with positive diagonal. Then, define the Poisson map $\Lambda : \mathcal{M} \to B(n) \equiv U(n)^*$ by

$$\Lambda(K, w^1, \ldots, w^d) := b_L b_R \mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d), \quad \text{with} \quad (w^1, w^2, \ldots, w^d) \in \mathbb{C}^{n\times d}.$$

This generates an action of the Poisson–Lie group $U(n)$ on $\mathcal{M}$, and we obtain the reduced phase space

$$\mathcal{M}_{\mathrm{red}} = \Lambda^{-1}(e^{\gamma}\mathbf{1}_n)/U(n),$$

which is a smooth symplectic manifold for any $\gamma > 0$.

The unreduced phase space carries the commuting Hamiltonians

$$H_j := \mathrm{tr}(L^j) \quad \text{with} \quad L := b_R b_R^\dagger, \quad j = 1, \ldots, n.$$

They have very simple flows and yield an integrable system on $\mathcal{M}$, quite similar to the cotangent bundle case.

We can go to the gauge slice where g_R becomes a diagonal matrix, $Q \in \mathbb{T}^n_{\rm reg}$. Decomposing $b \in B(n)$ as $b = b_0 b_+$, with diagonal and unipotent factors, we write

$$b_R = b_0 b_+ \quad \text{and} \quad \mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d) =: S(W) =: S_0(W) S_+(W).$$

Then the moment map condition becomes equivalent to the following constraints:

$$S_0(W) = e^{\gamma}\mathbf{1}_n \quad \text{and} \quad b_+ S_+(W) = Q^{-1} b_+ Q.$$

The first equation constraints $W = (w^1, \ldots, w^d)$ only, while the second one permits us to express b_+ in terms of $Q = e^{\mathrm{i}q} \in \mathbb{T}^n_{\rm reg}$ and $S_+(W) \in \mathbb{C}^{n\times d}$. (Same eq. as $b_+\lambda = Q^{-1} b_+ Q$.)

$Q \in \mathbb{T}^n_{\rm reg}$ and $b_0 \equiv \exp(p)$, with $p = \mathrm{diag}(p_1, \ldots, p_n)$, are arbitrary, and a dense open subset of the reduced phase space is parametrized by Q, p and the constrained 'primary spins', W, up to the usual residual gauge transformations.

The reduction of the spectral invariants of $L = b_R b_R^\dagger$ yields an integrable system.

To connect our reduced system with the Gibbons–Hermsen model, we introduce a positive ‘scaling parameter’ ϵ and make the replacements

$$p \to \epsilon p, \quad W \to \epsilon^{\frac{1}{2}} W, \quad Q \to Q, \quad \Omega_{\mathcal{M}} \to \epsilon^{-1} \Omega_{\mathcal{M}}, \quad \gamma \to \epsilon \gamma,$$

where $\Omega_{\mathcal{M}}$ is the symplectic form on $\mathcal{M}$. With $L := b_R b_R^\dagger$ and $b_R = \exp(\epsilon p) b_+(Q, \epsilon^{\frac{1}{2}} W)$, writing $Q = \mathrm{diag}(e^{\mathrm{i}q_1}, \dots, e^{\mathrm{i}q_n})$ and letting w_i denote the i-th row of $W \in \mathbb{C}^{n\times d}$, we get

$$\lim_{\epsilon\to 0} \frac{1}{8\epsilon^2}(\mathrm{tr}(L) + \mathrm{tr}(L^{-1}) - 2n) = \frac{1}{2}\mathrm{tr}(p^2) + \frac{1}{32}\sum_{i\neq j} \frac{|w_i^\alpha w_j^\dagger|^2}{\sin^2 \frac{q_i - q_j}{2}},$$

$$\lim_{\epsilon\to 0} \epsilon^{-1} (\Omega_{\mathrm{red}}) = \sum_{j=1}^{n} dp_j \wedge dq_j + \frac{\mathrm{i}}{2} \sum_{j=1}^{n} \sum_{\alpha=1}^{d} dw_j^\alpha \wedge d\overline{w}_j^\alpha,$$

reproducing the Hamiltonian and symplectic form of the Gibbons–Hermsen model.

Details are explained in Fairon, L.F. and Marshall: *Trigonometric real form of the spin RS model of Krichever and Zabrodin*, [arXiv:2007.08388].

Our construction is a ‘real form’ of earlier reduction treatments of the holomorphic spin RS models of Krichever–Zabrodin (1995), which are due to Chalykh and Fairon [arXiv:1811.08727] and to Arutyunov and Olivucci [arXiv:1906.02619]. The connection to the Gibbons–Hermsen model was not noticed in those papers.

We finish by sketching the degenerate integrability of the reduced system. For this, we exhibit the sufficient number of integrals of motion. To do this, we introduce the new 'spin vectors' $v(1),\dots,v(d)$ with $v(\alpha) := b_R \mathbf{b}(w^1)\cdots\mathbf{b}(w^{\alpha-1})w^\alpha$, which transform nicely under $\mathrm{U}(n)$. Then, we consider the polynomial subalgebra of $C^\infty(\mathcal{M})^{\mathrm{U}(n)}$:

$$\mathcal{I}_L = \mathbb{R}[\mathrm{tr} L^k, \Re(I^k_{\alpha\beta}), \Im(I^k_{\alpha\beta}) \mid 1 \le \alpha,\beta \le d,\ k \ge 0],\quad I^k_{\alpha\beta} := \mathrm{tr}\left(v(\alpha)v(\beta)^\dagger L^k\right).$$

This is closed under the Poisson bracket and its center contains

$$\mathfrak{H}_{\mathrm{tr}} := \mathbb{R}[\mathrm{tr} L^k, k \ge 0]\,.$$

Explicitly, we have

$$\begin{aligned}\{I^M_{\alpha\beta}, I^N_{\gamma\epsilon}\} = {} & 2\mathrm{i}\delta_{\alpha\epsilon} I^{M+N+1}_{\gamma\beta} - 2\mathrm{i}\delta_{\gamma\beta} I^{M+N+1}_{\alpha\epsilon} \\ & + \mathrm{i}(\delta_{\alpha\epsilon} - \delta_{\gamma\beta}) I^M_{\alpha\beta} I^N_{\gamma\epsilon} + 2\mathrm{i}\delta_{\alpha\epsilon} \sum_{\mu<\alpha} I^N_{\gamma\mu} I^M_{\mu\beta} - 2\mathrm{i}\delta_{\gamma\beta} \sum_{\lambda<\beta} I^M_{\alpha\lambda} I^N_{\lambda\epsilon} \\ & + \mathrm{i}\,\mathrm{sgn}(\gamma-\alpha) I^M_{\gamma\beta} I^N_{\alpha\epsilon} - \mathrm{i}\,\mathrm{sgn}(\epsilon-\beta) I^N_{\gamma\beta} I^M_{\alpha\epsilon} \\ & + \mathrm{i}\left(\sum_{b=0}^{M-1} + \sum_{b=0}^{N-1}\right)\left(I^b_{\gamma\beta} I^{M+N-b}_{\alpha\epsilon} - I^{M+N-b}_{\gamma\beta} I^b_{\alpha\epsilon}\right)\end{aligned}$$

and the reality property $\{\overline{I^M_{\alpha\beta}}, \overline{I^N_{\gamma\epsilon}}\} = \overline{\{I^M_{\alpha\beta}, I^N_{\gamma\epsilon}\}}$.

Our Hamiltonian reduction actually works in the real-analytic category, and $\mathfrak{H}_{\mathrm{tr}}$ and $\mathcal{I}_L$ descend to polynomial Poisson algebras on the connected, real-analytic reduced symplectic manifold $(\mathcal{M}_{\mathrm{red}}, \Omega_{\mathrm{red}})$.

Theorem. The reduced polynomial algebras of functions $\mathfrak{H}_{\mathrm{tr}}^{\mathrm{red}}$ and $\mathcal{I}_L^{\mathrm{red}}$ inherited from $\mathfrak{H}_{\mathrm{tr}}$ and $\mathcal{I}_L$ have functional dimension n and $2nd - n$, respectively. In particular, on the phase space $\mathcal{M}_{\mathrm{red}}$ of dimension $2nd$, the Abelian Poisson algebra $\mathfrak{H}_{\mathrm{tr}}^{\mathrm{red}}$ yields a real-analytic, degenerate integrable system with integrals of motion $\mathcal{I}_L^{\mathrm{red}}$

Concretely, for any $d > 1$, we proved that the $2n(d-1)$ integrals of motion:

$$\mathrm{tr}(L^k), \quad I^k_{1,1}, \quad \Re[I^k_{\alpha,1}], \quad \Im[I^k_{\alpha,1}]$$

with $k = 1, \ldots, n$ and $\alpha = 2, \ldots, d-1$, are independent after reduction, and n further integrals of motion may be selected from the real and imaginary parts of the functions $I^k_{d,1}$ in such a way that all in all these provide a set of $2nd - n$ independent functions.

In the $d = 1$ case $\mathfrak{H}_{\mathrm{tr}}^{\mathrm{red}} = \mathcal{I}_L^{\mathrm{red}}$ and one has (only) Liouville integrability.

Further results about Poisson reduction. Identity the phase space as

$$\mathcal{M} = \{(g, L, v)\} = \mathrm{U}(n) \times \mathfrak{P}(n) \times \mathbb{C}^{n\times d}, \qquad \mathfrak{P}(n) := \exp(\mathrm{i}\mathfrak{u}(n)),$$

and consider the U(n)-**equivariant Poisson map**

$$\Psi : \mathcal{M} \to \mathfrak{P}(n) \times \mathfrak{P}(n) \times \mathbb{C}^{n\times d}, \quad \Psi(g, L, v) := (g^{-1}Lg, L, v),$$

where $v := (v(1), \ldots, v(d))$ with $v(\alpha) := b_R \mathbf{b}(w^1) \cdots \mathbf{b}(w^{\alpha-1}) w^{\alpha}$.
This encodes the constants of motion for the free system on $\mathcal{M}$.
Define $\mathfrak{C} := \Psi(\mathcal{M})$ and let $\mathfrak{C}_{\mathrm{reg}} \subset \mathfrak{C}$ be the subset where L is regular. Next, let $\mathfrak{C}_* \subset \mathfrak{C}_{\mathrm{reg}}$ and $\mathcal{M}_* \subset \mathcal{M}$ be the dense open subsets of principal orbit type. Then,

$$\mathcal{M}_{**} := \Psi^{-1}(\mathfrak{C}_*) \subset \mathcal{M}_*$$

is dense and open. We get degenerate integrability of the reduced free system on the Poisson quotient $\mathcal{M}_*/\mathrm{U}(n)$ using the restriction of the invariants $\Psi^*(C^\infty(\mathfrak{P}(n) \times \mathfrak{P}(n) \times \mathbb{C}^{n\times d})^{\mathrm{U}(n)})$ on $\mathcal{M}_*$. We can also prove degenerate integrability on the maximal symplectic leaves of $\mathcal{M}_{**}/\mathrm{U}(n)$.

New results about multi-Hamiltonian structure. Let us consider the space of 'primary spins'

$$\mathcal{W} := \mathbb{C}^{n\times d} = \{(w^1, w^2, \dots, w^d)\},$$

and define on it the commuting vector fields V_j ($j = 1, \dots, d$) that as derivations of the evaluation functions satisfy

$$V_j[w^k] = \mathrm{i}\delta_{j,k} w^k.$$

They are the infinitesimal generators of the natural U(1) actions on the d-copies of $\mathbb{C}^n$. They are naturally extended to $\mathcal{M} = \mathrm{GL}(n, \mathbb{C}) \times \mathcal{W}$, and the previously introduced Poisson bivector $P_{\mathcal{M}}$ admits the modification

$$P_{\mathcal{M}} \to P_{\mathcal{M}} + \sum_{1\leq j<k\leq d} x_{jk} V_j \wedge V_k$$

with arbitrary real parameters x_{jk}. The modified Poisson structure remains symplectic. It admits the same Poisson–Lie moment map as for $x_{jk} \equiv 0$, generating the same $\mathrm{U}(n)$ action, and the flows of the free Hamiltonians do not change.

As a result, we obtain a multi-Hamiltonian structure for the reduced free system on $\mathcal{M}_*/\mathrm{U}(n)$, and on every symplectic leaf thereof. (One may also study the full Poisson quotient space $\mathcal{M}/\mathrm{U}(n)$.)