Integrable Hamiltonian systems from reductions of doubles of compact Lie groups I

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To begin, recall that the classical Sutherland Hamiltonian, with coupling constant x^2,

$$H_{\text{trig}-\text{Suth}}(q,p) \equiv \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{8}\sum_{j\neq k} \frac{x^2}{\sin^2((q_j-q_k)/2)},$$

admits two kinds of spin extensions. The first one contains Lie algebraic ('collective') spin variables,

$$H_{\text{spin}-\text{Suth}}(q,p,\xi) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{8}\sum_{j\neq k} \frac{|\xi_{jk}|^2}{\sin^2((q_j-q_k)/2)},$$

where $\xi \in \mathfrak{u}(n)^*$, with zero diagonal part. These models exist for all simple Lie algebras,

$$H_{\text{spin}-\text{Suth}}(q,p,\xi) = \frac{1}{2}\langle p,p\rangle + \frac{1}{8}\sum_{\alpha\in\mathfrak{R}} \frac{2}{|\alpha|^2}\frac{|\xi_\alpha|^2}{\sin^2(\alpha(q)/2)},$$

and arise from Hamiltonian reduction of the cotangent bundle T^*G of a compact Lie group G. The 'spin variables' $\xi_\alpha \in \mathbb{C}$ $(\xi_{-\alpha} = \xi_\alpha^*)$ matter up to gauge transformation by the maximal torus $G_0 < G$ and $q,p \in \mathrm{i}\mathcal{G}_0$ with $\mathcal{G}_0 = \mathrm{Lie}(G_0)$. Here, we use the Killing form and the set of roots $\mathfrak{R} = \{\alpha\}$ of the complexified Lie algebra $\mathcal{G}^{\mathbb{C}}$.

The second kind of generalization is the Gibbons–Hermsen model

$$H_{\text{G-H}} = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \frac{1}{8}\sum_{j\neq k} \frac{|(S_j S_k^\dagger)|^2}{\sin^2((q_j - q_k)/2)}.$$

The complex row-vector $S_j := [S_{j1}, \ldots, S_{jd}] \in \mathbb{C}^d$, $d \geq 2$, is attached to the particle with coordinate q_j, representing internal degrees of freedom. The overall phases of the spin vectors S_j can be changed by gauge transformations. This model descends from the extended cotangent bundle $T^*U(n) \times \mathbb{C}^{n\times d}$.

There exist 'relativistic' generalizations of the trigonometric Sutherland models , namely the Ruijsenaars–Schneider models: like

$$H_{\text{trig-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j\neq k} \left[1 + \frac{\sinh^2 x}{\sin^2 (q_j - q_k)/2}\right]^{\frac{1}{2}}$$

$$H_{\text{compact-RS}} = \sum_{k=1}^{n} (\cos p_k) \sqrt{\prod_{j\neq k} \left[1 - \frac{\sin^2 x}{\sin^2 (q_j - q_k)/2}\right]},$$

which also admit spin extensions. Of course, there exist also hyperbolic versions and rational degenerations, like the original Calogero–Moser model,

$$H_{\text{CM}}(q,p) \equiv \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{(q_j - q_k)^2)},$$

and elliptic generalizations, but those will not be treated in this mini-course.

In this first lecture, after some generalities, I shall focus on the derivation of the Lie algebraic spin Sutherland models and on explain how their integrability can be understood in the framework of Hamiltonian reduction. The second and third lectures will be devoted to Ruijsenaars–Schneider type deformations of the spin Sutherland systems, and to the compactified Ruijsenaars–Schneider model, respectively.

Plan of the lecture

1. The notion of 'integrable system'

2. The three doubles and their master integrable systems

3. The simplest example: Spin Sutherland models from cotangent bundles

4. The dual system in a nutshell

3. The notion of generalized action variables and action-angle coordinates

5. Action variables in our examples and their application

Today's lecture is mainly based on the following papers, where one can find references regarding, for example, the previously mentioned integrable many-body models.

• LF, *Poisson reductions of master integrable systems on doubles of compact Lie groups*; arXiv:2208.03728

• LF, *Notes on the degenerate integrability of reduced systems obtained from the master systems of free motion on cotangent bundles of compact Lie groups*, arXiv:2309.16245

We deal with classical integrable systems adopting the following **definition**.

Let $(\mathcal{M}, P_{\mathcal{M}})$ be a finite dimensional, connected, C^∞ Poisson manifold, and $\mathfrak{H}$ an Abelian Poisson subalgebra of $C^\infty(\mathcal{M})$ subject to the conditions:
1. As a commutative algebra of functions $\mathfrak{H}$ has functional dimension $\mathrm{ddim}(\mathfrak{H}) = \ell$.
2. The Hamiltonian vector fields of the elements of $\mathfrak{H}$ are complete and span an ℓ dimensional subspace of the tangent space over a dense open subset of $\mathcal{M}$.
3. The commutant $\mathfrak{F}$ of $\mathfrak{H}$ in $C^\infty(\mathcal{M})$, which contains the joint constants of motion of the Hamiltonians $\mathcal{H} \in \mathfrak{H}$, has functional dimension $\mathrm{ddim}(\mathfrak{F}) = \mathrm{dim}(\mathcal{M}) - \ell$.

We refer to the quadruple $(\mathcal{M}, P_{\mathcal{M}}, \mathfrak{H}, \mathfrak{F})$, or simply $\mathfrak{H}$, as a *(degenerate) integrable system of rank* ℓ. The standard notion of Liouville integrability results if $\mathcal{M}$ is a symplectic manifold and $\ell = \mathrm{dim}(\mathcal{M})/2$. Liouville integrability on Poisson manifolds is the case for which $\ell = k$, where k is half the dimension of the maximal symplectic leaves. When $\ell < k$, both on Poisson and symplectic manifolds, then one obtains the case of degenerate integrability, alternatively called superintegrability. A single Hamiltonian is called (super)integrable if it is a member of $\mathfrak{H}$ obeying the definition.

Standard example. Take the Kepler–Coulomb problem governed by the Hamiltonian $H(\vec{r}, \vec{p}) = p^2/2m - \gamma/r$ and canonical Poisson brackets. Let $\mathfrak{H}$ be generated by H. Then $\mathfrak{F}$ is generated by H, the angular momentum $\vec{L} = \vec{r} \wedge \vec{p}$ and the Runge–Lenz vector $\vec{K} = \vec{p} \wedge \vec{L} - \gamma m \vec{r}/r$. One account of the relations $\vec{L} \cdot \vec{K} = 0$ and $K^2 = m^2\gamma^2 + 2mL^2H$, one has $\mathrm{ddim}(\mathfrak{F}) = 5$, and the system is 'superintegrable'.

The three doubles and the general philosophy

Let G be a (connected and simply connected) compact Lie group with simple Lie algebra $\mathcal{G}$. Denote $\mathcal{G}^{\mathbb{C}}$ and $G^{\mathbb{C}}$ the complexifications, and define $\mathfrak{P} := \exp(\mathrm{i}\mathcal{G}) \subset G^{\mathbb{C}}$. Example: $G = SU(n)$, $G^{\mathbb{C}} = SL(n, \mathbb{C})$, $\mathfrak{P} = \{X \in SL(n, \mathbb{C}) \mid X^{\dagger} = X,\ X \text{ positive}\}$.

One has the following 3 ‘classical doubles’ of G:

Cotangent bundle $\quad T^*G \simeq G \times \mathcal{G}^* \simeq G \times \mathcal{G} =: \mathcal{M}_1$

Heisenberg double $\quad G^{\mathbb{C}}_{\mathbb{R}} \simeq G \times G^* \simeq G \times \mathfrak{P} =: \mathcal{M}_2$

Internally fused quasi-Poisson double $\quad G \times G =: \mathcal{M}_3$

The pull-backs of the relevant rings of invariants

$$C^\infty(G)^G, \quad C^\infty(\mathcal{G})^G, \quad C^\infty(\mathfrak{P})^G$$

give rise to two ‘master integrable systems’ on each double.

The group G acts on these phase spaces by ‘diagonal conjugations’, i.e., by the diffeomorphisms

$$A^i_\eta : (x, y) \mapsto (\eta x \eta^{-1}, \eta y \eta^{-1}), \quad \forall (x, y) \in \mathcal{M}_i \ (i = 1, 2, 3),\ \eta \in G.$$

The G-invariant functions form closed Poisson algebras, and thus the quotient space $\mathcal{M}_i^{\text{red}} \equiv \mathcal{M}_i/G$ becomes a (singular) Poisson space, which carries the corresponding reduced integrable systems.

A degenerate integrable system on the cotangent bundle T^*G

The canonical Poisson bracket on the cotangent bundle

$$\mathcal{M} \equiv T^*G \simeq G \times \mathcal{G}^* \simeq G \times \mathcal{G} = \{(g, J) \mid g \in G,\ J \in \mathcal{G}\} \text{ has the form}$$

$$\{\mathcal{F}, \mathcal{H}\}(g, J) = \langle \nabla_1 \mathcal{F}, d_2\mathcal{H} \rangle - \langle \nabla_1 \mathcal{H}, d_2\mathcal{F} \rangle + \langle J, [d_2\mathcal{F}, d_2\mathcal{H}] \rangle,$$

where the $\mathcal{G}$-valued derivatives are taken at (g, J). Here, $\langle X, Y \rangle$ is the Killing form on $\mathcal{G}$. The derivative $d_2\mathcal{F} \in \mathcal{G}$ w.r.t. the second variable $J \in \mathcal{G}$ is the usual gradient, while the derivative $\nabla_1 \mathcal{F} \in \mathcal{G}$ w.r.t. first variable $g \in G$ is defined by

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}(e^{tX} g, J) =: \langle X, \nabla_1 \mathcal{F}(g, J) \rangle, \quad \forall X \in \mathcal{G}.$$

The equations of motion generated by the Hamiltonians $\mathcal{H}$ of the form $\mathcal{H}(g, J) = \varphi(J)$ with $\varphi \in C^\infty(\mathcal{G})^G$ read

$$\dot{g} = (d\varphi(J))g,\ \dot{J} = 0 \implies (g(t), J(t)) = (\exp(td\varphi(J(0)))g(0), J(0)).$$

The constants of motions are arbitrary functions of J and $\tilde{J} := g^{-1}Jg$.

We get a degenerate integrable system with

$$\mathfrak{H} := \{\mathcal{H} \mid \mathcal{H}(g, J) = \varphi(J),\ \varphi \in C^\infty(\mathcal{G})^G\}, \quad \mathrm{ddim}(\mathfrak{H}) = \mathrm{rank}(\mathcal{G}) := \ell$$

$$\mathfrak{F} : \text{arbitrary smooth functions of } J \text{ and } \tilde{J}, \qquad \mathrm{ddim}(\mathfrak{F}) = 2\mathrm{dim}(\mathcal{G}) - \ell,$$

since J and $\tilde{J}$ are related by the ℓ constraints $P_i(J) = P_i(\tilde{J}) = 0$, where the P_i $(i = 1, \ldots, \ell)$ are independent invariant polynomials on $\mathcal{G}$.

We call this ‘the integrable system of free motion’.

The group G acts on $\mathcal{M}$ by diagonal conjugations, i.e., by the following maps:

$$A_\eta : (g, J) \mapsto (\eta g \eta^{-1}, \eta J \eta^{-1}, \quad \forall \eta \in G, \quad \text{with} \quad \eta J \eta^{-1} := \mathrm{Ad}_\eta(J).$$

This is a Hamiltonian action, with the 'momentum map' $\Phi(g, J) = J - g^{-1}Jg = J - \tilde{J}$. The elements of $\mathfrak{H}$ and the Poisson structure are G-invariant.

Hamiltonian reduction means that we keep only the invariant with respect to a symmetry group. Geometrically, this amounts 'projecting' the system onto the quotient space of the phase space with respect to the action of the symmetry group. A technical difficulty is that the quotient space $\mathcal{M}/G$ is not a smooth manifold.

Let $\mathcal{G}_0 < \mathcal{G}$ a maximal Abelian subalgebra and $G_0 = \exp(\mathcal{G}_0) < G$ the maximal torus. Denote $G^{\mathrm{reg}} \subset G$ the dense open subset of regular elements, and put $G_0^{\mathrm{reg}} := G^{\mathrm{reg}} \cap G_0$. Here, G^{reg} contains the group elements whose centralizer is a maximal torus.

Now, we characterize the reduced system using a 'partial gauge fixing'. Define

$$\mathcal{M}^{\mathrm{reg}} := \{(g, J) \in \mathcal{M} \mid g \in G^{\mathrm{reg}}\}, \quad \mathcal{M}_0^{\mathrm{reg}} := \{(Q, J) \in \mathcal{M} \mid Q \in G_0^{\mathrm{reg}}\}.$$

The normalizer $\mathfrak{N}$ of $G_0 < G$ serves as the 'group of residual gauge transformations'.

Then, $\mathcal{M}^{\mathrm{reg}}/G \equiv \mathcal{M}_0^{\mathrm{reg}}/\mathfrak{N}$, and the restriction of functions yields the isomorphism

$$C^\infty(\mathcal{M}^{\mathrm{reg}})^G \Longleftrightarrow C^\infty(\mathcal{M}_0^{\mathrm{reg}})^{\mathfrak{N}}.$$

Thus, we can transfer the Poisson bracket from $C^\infty(\mathcal{M}^{\mathrm{reg}})^G$ to $C^\infty(\mathcal{M}_0^{\mathrm{reg}})^{\mathfrak{N}}$. For any $F, H \in C^\infty(\mathcal{M}_0)^{\mathfrak{N}}$, the resulting 'reduced Poisson bracket' is defined by

$$\{F, H\}_{\mathrm{red}}(Q, J) = \{\mathcal{F}, \mathcal{H}\}(Q, J),$$

where F, H are the restrictions of $\mathcal{F}, \mathcal{H} \in C^\infty(\mathcal{M}^{\mathrm{reg}})^G$ onto the 'gauge slice' $\mathcal{M}_0^{\mathrm{reg}}$.

To describe the result, consider the decomposition $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_\perp$, where $\mathcal{G}_\perp$ is the orthogonal complement of $\mathcal{G}_0$. Define $\mathcal{R}(Q) \in \mathrm{End}(\mathcal{G})$ that vanishes on $\mathcal{G}_0$ and, writing $Q = \exp(\mathrm{i}q)$ with $\mathrm{i}q \in \mathcal{G}_0$, is given on $\mathcal{G}_\perp$ by $\mathcal{R}(Q) = \frac{1}{2}\coth(\frac{\mathrm{i}}{2}\mathrm{ad}_q)$. For $SU(n)$, $(\mathcal{R}(Q)X)_{jk} = \frac{1}{2}(1-\delta_{jk})X_{jk}\coth(\frac{\mathrm{i}}{2}(q_j - q_k))$. Defining $[X,Y]_\mathcal{R} \equiv [\mathcal{R}X, Y] + [X, \mathcal{R}Y]$, $\forall X, Y \in \mathcal{G}$, the result is:

$$\{F,H\}_{\mathrm{red}}(Q,J) = \langle \nabla_1 F, d_2 H\rangle - \langle \nabla_1 H, d_2 F\rangle + \langle J, [d_2 F, d_2 H]_{\mathcal{R}(Q)}\rangle.$$

Up to residual gauge transformations, the 'reduced evolution equation' generated on $\mathcal{M}_0^{\mathrm{reg}}$ by $H(Q,J) = \varphi(J)$, with $\varphi \in C^\infty(\mathcal{G})^G$, can be written as

$$\dot{Q} = (d\varphi(J))_0 Q, \qquad \dot{J} = [\mathcal{R}(Q)d\varphi(J), J].$$

We can parametrize $J \in \mathcal{G}$ according to

$$J = -\mathrm{i}p + \sum_{\alpha\in\mathfrak{R}_+}\left(\frac{\xi_\alpha}{e^{-\mathrm{i}\alpha(q)}-1}e_\alpha - \frac{\xi_\alpha^*}{e^{\mathrm{i}\alpha(q)}-1}e_{-\alpha}\right), \qquad p \in \mathrm{i}\mathcal{G}_0.$$

Then, we obtain $$-\frac{1}{2}\langle J, J\rangle = \frac{1}{2}\langle p,p\rangle + \frac{1}{8}\sum_{\alpha\in\mathfrak{R}}\frac{2}{|\alpha|^2}\frac{|\xi_\alpha|^2}{\sin^2(\alpha(q)/2)},$$

which is a standard spin Sutherland Hamiltonian $H_{\mathrm{spin-Suth}}(q,p,\xi)$. Here, we use the root space decomposition of the complexified Lie algebra $\mathcal{G}^\mathbb{C}$, with the set of roots $\mathfrak{R} = \{\alpha\}$ and corresponding root vectors $e_\alpha \in \mathcal{G}_\perp^\mathbb{C}$.

The 'spin variable' $\xi = \sum_{\alpha\in\mathfrak{R}_+}(\xi_\alpha e_\alpha - \xi_\alpha^* e_{-\alpha})$ matters up to conjugations by any $T \in G_0$. After dividing by G_0, there remains a residual gauge symmetry under the Weyl group $W = \mathfrak{N}/G_0$. The pertinent dense open subset of the reduced phase space can be identified as $\left(T^*G_0^{\mathrm{reg}} \times (\mathcal{G}^*//_0 G_0)\right)/W$, with Darboux variables (q,p) on $T^*G_0^{\mathrm{reg}} \simeq G_0^{\mathrm{reg}} \times \mathcal{G}_0$ and spin variable $[\xi] \in \mathcal{G}^*//_0 G_0$.

We have identified the reduced Hamiltonian coming from the 'kinetic energy' as the spin Sutherland Hamiltonian, at least on a dense open subset of the quotient space. What about the integrability of the reduced system?

Let $\mathfrak{F}_{\mathrm{red}}$ denote the constants of motion of the reduced Abelian Poisson algebra $\mathfrak{H}_{\mathrm{red}}$. Since $\mathfrak{H}_{\mathrm{red}}$ contains all invariant functions of J, we have $\mathrm{ddim}(\mathfrak{H}_{\mathrm{red}}) = \mathrm{ddim}(\mathfrak{H}) = \ell$.

We wish to show that $\mathrm{ddim}(\mathfrak{H}_{\mathrm{red}}) + \mathrm{ddim}(\mathfrak{F}_{\mathrm{red}}) = \mathrm{ddim}(\mathcal{M}/G)$. To do so, consider the map $\Psi : (g, J) \mapsto (\tilde{J}, J)$ and let $\mathfrak{C} \subset \mathcal{G} \times \mathcal{G}$ be its image. This map is G-equivariant, and we get the commutative diagram:

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\quad\Psi\quad} & \mathfrak{C} \\
{\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\
\mathcal{M}/G & \xrightarrow{\quad\Psi_{\mathrm{red}}\quad} & \mathfrak{C}/G
\end{array}$$

Since Ψ is constant along the integral curves of $\mathcal{H} \in \mathfrak{H}$, Ψ_{red} is constant along the integral curves of $\mathcal{H}_{\mathrm{red}} \in \mathfrak{H}_{\mathrm{red}}$. Therefore, $\Psi^*_{\mathrm{red}} C^\infty(\mathfrak{C}/G)$ gives constants of motion for the reduced system. In favourable circumstances, we get

$$\mathrm{ddim}\left(\Psi^*_{\mathrm{red}} C^\infty(\mathfrak{C}/G)\right) = \dim(\mathfrak{C}/G) = \dim(\mathfrak{C}) - \dim(G) = \mathrm{ddim}(\mathfrak{F}) - \dim(G),$$

and $\dim(\mathcal{M}/G) = \dim(M) - \dim(G)$. Putting this together, we obtain

$$\dim(\mathcal{M}) = \mathrm{ddim}(\mathfrak{H}) + \mathrm{ddim}(\mathfrak{F}) \Longrightarrow \dim(\mathcal{M}/G) = \mathrm{ddim}(\mathfrak{H}_{\mathrm{red}}) + \mathrm{ddim}(\mathfrak{F}_{\mathrm{red}}).$$

This implies degenerate integrability of the reduced system.

There is technical problem: neither $\mathcal{M}/G$ nor $\mathfrak{C}$ are smooth manifolds. For this reasen, we restrict to the dense open submanifold $\mathcal{M}_* \subset \mathcal{M}$ of principal orbit type.

It is easily seen $\mathcal{M}_* = \{(g, J) \in \mathcal{M} \mid G_{(g,J)} = \mathcal{Z}(G)\}$, and $\mathcal{M}_*/G$ is smooth. Moreover, $\mathcal{M}_*$ is invariant with respect to the Hamiltonian flow of any $\mathcal{F} \in C^\infty(\mathcal{M})^G$.

Next, we introduce $\mathfrak{C}_{\mathrm{reg}} := \{\tilde{J}, J) \in \mathfrak{C} \mid J \in \mathcal{G}^{\mathrm{reg}}\}$, which is a smooth, embedded submanifold of $\mathcal{G}^{\mathrm{reg}} \times \mathcal{G}^{\mathrm{reg}}$.

A key point to introduce also

$$\mathfrak{C}_* := \{(\tilde{J}, J) \in \mathfrak{C}_{\mathrm{reg}} \mid G_{(\tilde{J},J)} = \mathcal{Z}(G)\} \quad \text{and} \quad \mathcal{M}_{**} := \Psi^{-1}(\mathfrak{C}_*).$$

These are dense open submanifolds of $\mathfrak{C}_{\mathrm{reg}}$ and of $\mathcal{M}_*$. The restriction of Ψ yields the G-equivariant submersion $\psi : \mathcal{M}_{**} \to \mathfrak{C}_*$, and we get the diagram of **smooth** Poisson submersions (where $\mathcal{M}_{**}^{\mathrm{red}} = \mathcal{M}_{**}/G$ and $\mathfrak{C}_*^{\mathrm{red}} = \mathfrak{C}_*/G$):

$$\begin{array}{ccc} \mathcal{M}_{**} & \xrightarrow{\ \psi\ } & \mathfrak{C}_* \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathcal{M}_{**}^{\mathrm{red}} & \xrightarrow{\ \psi_{\mathrm{red}}\ } & \mathfrak{C}_*^{\mathrm{red}} \end{array}$$

Now the previous calculation goes through rigorously, and we obtain a degenerate integrable system on the dense open submanifold $\mathcal{M}_{**}^{\mathrm{red}} \subset \mathcal{M}_*^{\mathrm{red}}$ (which is stable under the flows of the commuting reduced Hamiltonians).

A slight extension of the method shows integrability on the full smooth Poisson manifold $\mathcal{M}_*^{\text{red}} = \mathcal{M}_*/G$.

One can also prove that the Hamiltonian vector fields of $\mathfrak{H}_{\text{red}}$ span an ℓ-dimensional subspace of the tangent space at every point of $\mathcal{M}_{**}^{\text{red}}$. In general, $\mathcal{M}_{**}^{\text{red}} \subset \mathcal{M}_*^{\text{red}}$ is a proper subset.

Consider ℓ independent invariant polynomials, P_i $(i = 1, \ldots, \ell)$ on $\mathcal{G}$. The functions $P_i \circ \Phi \in C^\infty(\mathcal{M})^G$ (where Φ is the momentum map) yield the center of $C^\infty(\mathcal{M})^G$. Fixing these Casimir functions defines symplectic leaves in $\mathcal{M}_{**}^{\text{red}}$ This entails that degenerate integrability holds after restriction on the generic symplectic leaves of $\mathcal{M}_{**}^{\text{red}}$, of co-dimension ℓ, too.

In fact, fixing the Casimir functions $P_i \circ \Phi$ decreases the number of independent constants of motion by ℓ, but it also decreases the dimension of the phase space by ℓ. The restriction of $\mathfrak{H}_{\text{red}}$ has functional dimension ℓ on every symplectic leaf of $\mathcal{M}_{**}^{\text{red}}$.

The method presented above is a refinement of the method applied by N. Reshetikhin in: *Degenerate integrability of spin Calogero–Moser systems and the duality with the spin Ruijsenaars systems, Lett. Math. Phys.* **63** *(2003) 55-71*; arXiv:math/0202245

The other degenerate integrable system on $\mathcal{M} = T^*G$ is defined by the Abelian Poisson algebra

$$\tilde{\mathfrak{H}} := \left\{ \mathcal{H} \mid \mathcal{H}(g, J) = h(g),\ h \in C^\infty(G)^G \right\}.$$

This again has functional dimension $\ell = \mathrm{rank}(\mathcal{G})$. The Hamiltonian $\mathcal{H}$ induces the dynamics

$$\dot{g} = 0,\ \dot{J} = -\nabla h(g), \quad (g(t), J(t)) = (g(0), J(0) - t\nabla h(g(0)).$$

The constants of motion are arbitrary functions of the pair (g, Φ), with the momentum map $\Phi(g, J) = J - \tilde{J}$. They form a Poisson algebra $\tilde{\mathfrak{F}}$ of functional dimension $2\mathrm{dim}(G) - \ell$, since Φ satisfies

$$\langle X, \Phi(g, J) \rangle = 0, \quad \forall X \in \mathcal{G} \quad \text{for which} \quad gXg^{-1} = X,$$

and this gives ℓ independent constraints generically, i.e.., if $g \in G^{\mathrm{reg}}$.

To characterize the reduced system, we now introduce an other dense open subset and alternative gauge slice

$$\tilde{\mathcal{M}}^{\mathrm{reg}} := \{(g, J) \in \mathcal{M} \mid J \in \mathcal{G}^{\mathrm{reg}}\}, \quad \tilde{\mathcal{M}}_0^{\mathrm{reg}} := \{(g, \lambda) \in \mathcal{M} \mid \lambda \in \mathcal{G}_0^{\mathrm{reg}}\}.$$

On this gauge slice, we find the 'reduced Poisson bracket'

$$\{F,H\}^{\tilde{}}_{\text{red}}(g,\lambda) = \langle \nabla_1 F, d_2 H\rangle - \langle \nabla_1 H, d_2 F\rangle + \langle \nabla'_1 F, r(\lambda)\nabla'_1 H\rangle - \langle \nabla_1 F, r(\lambda)\nabla_1 H\rangle,$$

and the 'reduced evolution equations'

$$\dot{\lambda} = -(\nabla h(g))_0, \qquad \dot{g} = [g, r(\lambda)\nabla h(g)].$$

Here, $r(\lambda) \in \text{End}(\mathcal{G})$ is the standard rational dynamical r-matrix:

$$r(\lambda)X = ((\text{ad}\lambda)_{|\mathcal{G}_\perp})^{-1}(X_\perp), \quad \forall X = (X_0 + X_\perp) \in (\mathcal{G}_0 + \mathcal{G}_\perp).$$

For $\mathcal{G} = \text{su}(n)$, $\lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$, and $(r(\lambda)X)_{jk} = X_{jk}/(\lambda_j - \lambda_k)$.

The degenerate integrability of the reduced systems can be shown similarly as in the previous case, now using the 'equivariant map of constants of motion' given by $\tilde{\Psi} : T^*G \ni (g, J) \mapsto (g, \Phi(g, J)) \in G \times \mathcal{G}$.

These systems are usually referred to as rational spin Ruijsenaars type systems, but their description is much less developed as for the spin Sutherland models.

For $G = \text{SU}(n)$, on a special symplectic leaf, the reduced system gives the so-called Ruijsenaars dual of the trigonometric Sutherland model, with the main Hamiltonian

$$\widetilde{H}_{\text{rat-RS}}(\lambda, \theta) = \sum_{k=1}^{n} (\cos\theta_k) \prod_{j\neq k} \left[1 - \frac{x^2}{(\lambda_k - \lambda_j)^2}\right]^{\frac{1}{2}},$$

which descends from the class function $h(g) = \Re\text{tr}(g)$, and canonical Poisson brackets. More precisely, this description is valid only on a dense open subset of the phase space.

A convenient notion of generalized action variables

Consider an integrable system of rank ℓ on a connected Poisson manifold $(\mathcal{M}, P_{\mathcal{M}})$ given by the Abelian Poisson algebra $\mathfrak{H}$. Suppose that we have ℓ smooth functions $H_1, \ldots, H_\ell$ on a connected dense open submanifold $\check{\mathcal{M}} \subset \mathcal{M}$ subject to the properties:
(i) The map $(H_1, \ldots, H_\ell) : \check{\mathcal{M}} \to \mathbb{R}^\ell$ is the momentum map for a proper and effective action of an ℓ-dimensional 'generalized torus' $\mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}$ on $\check{\mathcal{M}}$.
(ii) The restriction of the elements of $\mathfrak{H}$ on $\check{\mathcal{M}}$ can be expressed in terms of $H_1, \ldots, H_\ell$ and the span of the exterior derivatives of the elements of $\mathfrak{H}$ coincides with the span of the exterior derivatives $dH_1, \ldots, dH_\ell$ at every point of $\check{\mathcal{M}}$.
Then, we say that the functions $H_1, \ldots, H_\ell$ are generalized action variables on $\check{\mathcal{M}}$ for the integrable system $\mathfrak{H}$.

Semi-locally, in a neighbourhood of any principal orbit of $\mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}$, the generalized action variables are part of generalized action-angle coordinates. If $\mathcal{M}$ is symplectic with $\ell_2 = 0$, then this follows from a generalization of the classical Liouville–Arnold theorem due to Nekhoroshev, 1972]. In the Poisson case, with $\ell_2 = 0$, it follows from a similar result of [Laurent-Gengoux, Miranda and Vanhaecke, arXiv:0805.1679].

One usually considers action variables on open, not necessarily dense, subsets of $\mathcal{M}$. In fact, such variables can always be constructed in a neighbourhood of any connected component of a regular, compact level surface of $\mathfrak{H}$, as is shown in the above mentioned references. Our stronger notion is applicable in many examples.

Actions of compact Lie groups are automatically proper, and traditionally one talks about action variables only in the compact case.

Theorem [LF-Fairon]. Assume that $(\mathcal{M}, P_{\mathcal{M}}, \mathfrak{H})$ is an integrable system on a connected smooth Poisson manifold of dimension d that admits generalized action variables $H_1, \ldots, H_\ell$ on a connected dense open submanifold $\check{\mathcal{M}}$. Let y_0 be a point of $\check{\mathcal{M}}$ with trivial isotropy group for the generalized torus action, and put $p_i := H_i - H_i(y_0)$. Then, there exist a $\mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}$-invariant open neighbourhood $\mathcal{U} \subset \mathcal{M}$ around y_0 and functions $\theta_1, \ldots, \theta_\ell, z_1, \ldots, z_{d-2\ell} : \mathcal{U} \to \mathbb{R}$ that possess the following properties:

(i) The functions $(e^{\mathrm{i}\theta_1}, \ldots, e^{\mathrm{i}\theta_{\ell_1}}, \theta_{\ell_1+1}, \ldots, \theta_\ell, p_1, \ldots, p_\ell, z_1, \ldots, z_{d-2\ell})$ define a diffeomorphism $\mathcal{U} \longrightarrow (\mathrm{U}(1)^{\ell_1} \times \mathbb{R}^{\ell_2}) \times C_\epsilon^{d-\ell}$ for some $\epsilon > 0$, with $C_\epsilon^{d-\ell}$ denoting a hypercube of dimension $d - \ell$, and y_0 corresponds to $(e, 0)$.

(ii) The Poisson structure can be written in terms of these coordinates as

$$P_{\mathcal{M}}|_{\mathcal{U}} = \sum_{i=1}^{\ell} \frac{\partial}{\partial \theta_i} \wedge \frac{\partial}{\partial p_i} + \sum_{\substack{a,b=1 \\ a<b}}^{d-2\ell} f_{ab}(z) \frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_b},$$

for some smooth functions f_{ab} depending only on $z_1, \ldots, z_{d-2\ell}$.

Moreover, $\mathcal{U}$ can be chosen in such a manner that the 'action coordinates' p_i and 'transversal coordinates' z_a can be expressed in terms of restrictions of elements of the Abelian Poisson algebra $\mathfrak{H}$ and its constants of motion $\mathfrak{F}$, respectively. The dynamics generated by any $\mathcal{H} \in \mathfrak{H}$ becomes linear in these coordinates.

Towards action variables on T^*G: Lie algebraic preparations

Let $G_0 < G$ be a maximal torus, with Lie algebra $\mathcal{G}_0 < \mathcal{G}$. Let us realize $\mathcal{G}$ as

$$\mathcal{G} = \mathrm{span}_{\mathbb{R}}\{\mathrm{i}h_{\alpha_j},\ (e_\alpha - e_{-\alpha}),\ \mathrm{i}(e_\alpha + e_{-\alpha}) \mid \alpha_j \in \Delta,\ \alpha \in \mathfrak{R}^+\},$$

using a Weyl–Chevalley basis of $\mathcal{G}^{\mathbb{C}}$:

$$e_\alpha,\ e_{-\alpha},\ h_{\alpha_j} \quad \text{with} \quad \alpha \in \mathfrak{R}^+,\ j = 1,\ldots,\ell,$$

where $\Delta = \{\alpha_1,\ldots,\alpha_\ell\}$ is a base of the root system $\mathfrak{R}$ of $\mathcal{G}^{\mathbb{C}}$ with respect to $\mathcal{G}_0^{\mathbb{C}}$. Define the open Weyl chamber $\mathcal{C} \subset \mathrm{i}\mathcal{G}_0$ and the open Weyl alcove $\mathcal{A} \subset \mathcal{C} \subset \mathrm{i}\mathcal{G}_0$ as follows:

$$\mathcal{C} := \{X \in \mathrm{i}\mathcal{G}_0 \mid 0 < \alpha_j(X),\ \forall j = 1,\ldots,\ell\},$$

$$\mathcal{A} := \{X \in \mathrm{i}\mathcal{G}_0 \mid 0 < \alpha_j(X),\ \forall j = 1,\ldots,\ell,\ \text{and}\ \vartheta(X) < 2\pi\},$$

where ϑ is the highest root with respect to the base Δ. Then, introduce the *smooth* mappings $\underline{\varphi} : \mathcal{G}^{\mathrm{reg}} \to \mathcal{C}$ and $\underline{\chi} : G^{\mathrm{reg}} \to \mathcal{A}$ by the following recipes:

$$\underline{\varphi}(J) = \xi \quad \text{if} \quad \mathrm{i}\xi = \mathrm{Ad}_{\Gamma_1(J)}(J) \equiv \Gamma_1(J) J \Gamma_1(J)^{-1} \quad \text{for some} \quad \Gamma_1(J) \in G,$$

$$\underline{\chi}(g) = \xi \quad \text{if} \quad e^{\mathrm{i}\xi} = \Gamma_2(g) g \Gamma_2(g)^{-1} \quad \text{for some} \quad \Gamma_2(g) \in G.$$

The formulae

$$\varphi_j := \langle h_{\alpha_j}, \underline{\varphi}\rangle \quad \text{and} \quad \chi_j := \langle h_{\alpha_j}, \underline{\chi}\rangle, \qquad \forall j = 1,\ldots,\ell,$$

define real-analytic, K-invariant real functions φ_j on $\mathcal{G}^{\mathrm{reg}}$ and χ_j on G^{reg}, respectively. They can be extended to globally continuous functions, but not to smooth functions.

Let $\mathcal{P}_G$ and $\mathcal{P}_{\mathcal{G}}$ be the projections from $\mathcal{M} = T^*G \equiv G \times \mathcal{G}$ onto its respective factors.

Consider the G-invariant dense open, connected submanifolds of $\mathcal{M}$,

$$Y := G \times \mathcal{G}^{\text{reg}} \qquad \text{and} \qquad \tilde{Y} := G^{\text{reg}} \times \mathcal{G}.$$

(Note in passing that $Y \equiv \tilde{\mathcal{M}}^{\text{reg}}$ and $\tilde{Y} \equiv \mathcal{M}^{\text{reg}}$.) Using $H_j := \varphi_j \circ \mathcal{P}_{\mathcal{G}}$ and $\tilde{H}_j := \chi_j \circ \mathcal{P}_G$, fefine the G-invariant mappings

$$(H_1, \ldots, H_\ell) : Y \to \mathbb{R}^\ell \quad \text{and} \quad (\tilde{H}_1, \ldots, \tilde{H}_\ell) : \tilde{Y} \to \mathbb{R}^\ell.$$

Introduce the *diffeomorphisms* $\mathcal{T} : \mathbb{R}^\ell \to \mathcal{G}_0$ and $T : \mathbb{R}^\ell/(2\pi\mathbb{Z})^\ell \to G_0$ by

$$\mathcal{T}(\underline{\tau}) := -\mathrm{i}\sum_{j=1}^{\ell} \tau_j h_{\alpha_j} \quad \text{and} \quad T(\underline{\tau}) := \exp\big(-\mathrm{i}\sum_{j=1}^{\ell} \tau_j h_{\alpha_j}\big), \quad \forall \underline{\tau} = (\tau_1, \ldots, \tau_\ell) \in \mathbb{R}^\ell.$$

Lemma [LF-Fairon]. The map $(H_1, \ldots, H_\ell)$ is the momentum map for the free Hamiltonian action of the torus $\mathbb{T} := G_0$ on Y that works according to the formula

$$(T(\underline{\tau}), (g, J)) \mapsto (\Gamma_1(J)^{-1} T(\underline{\tau}) \Gamma_1(J) g, J), \quad \forall \underline{\tau} \in \mathbb{R}^\ell, \ (g, J) \in Y.$$

The map $(\tilde{H}_1, \ldots, \tilde{H}_\ell)$ serves as the momentum map generating the free and proper Hamiltonian action of $\mathbb{R}^\ell$ on $\tilde{Y}$ that operates as

$$(\underline{\tau}, (g, J)) \mapsto (g, J - \Gamma_2(g)^{-1} \mathcal{T}(\underline{\tau}) \Gamma_2(g)), \quad \forall \underline{\tau} \in \mathbb{R}^\ell, \ (g, J) \in \tilde{Y}.$$

These $\mathbb{T}$- and $\mathbb{R}^\ell$-actions commute with the G-actions restricted on Y and on $\tilde{Y}$. Over Y and $\tilde{Y}$, respectively, the elements of $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$ can be expressed as functions of the above momentum maps, which represent *generalized action variables*.

A new result regarding the T^*G example

Using the G-action, we apply Hamiltonian reduction to the Abelian Poisson algebras of the globally smooth Hamiltonians, $\mathfrak{H}$ and $\tilde{\mathfrak{H}}$, as well as to their action variables.

Let $\mathcal{M}_* \subset \mathcal{M}$ and $Y_0^1 \subset Y$ be the principal orbit type submanifolds for the G-action, and denote $Y_0 \subset Y$ the principal isotropy type submanifold for the action of $G \times \mathbb{T}$. These principal isotropy groups are $\mathcal{Z}(G)$ and $\mathcal{Z}(G) \times \{e\}$, respectively. We have $Y_0 \subset Y_0^1 \subset Y \subset \mathcal{M}$ and $Y_0^1 \subset \mathcal{M}_*$.

Theorem [LF-Fairon]. The Abelian Poisson algebra $\mathfrak{H}$ descends to an integrable system of rank ℓ on the Poisson manifold $\mathcal{M}_*/G$. The restrictions of this system to the Poisson manifolds Y_0/G and Y_0^1/G possess *action variables* given by $(H_1, \ldots, H_\ell)$, and the corresponding Hamiltonian $\mathbb{T}$-action is free on Y_0/G. As a result, $\mathfrak{H}$ induces integrable systems of rank ℓ with action variables arising from $(H_1, \ldots, H_\ell)$

- on every sympectic leaf $S \subset Y_0/G$;
- and on every such symplectic leaf $\mathcal{S} \subset Y_0^1/G$ that intersects Y_0/G.

The same statements hold if we replace $(Y, \mathfrak{H}, G\times\mathbb{T})$ by $(\tilde{Y}, \tilde{\mathfrak{H}}, G\times\mathbb{R}^\ell)$. Except for $\ell = 1$ and a few very small symplectic leaves, all these reduced systems are superintegrable.

This theorem a stronger than the previous results, since it establishes integrability on arbitrary symplectic leaves. Incidentally, it can be shown that $\mathcal{M}_{**} \subseteq Y_0$ holds.

The action-angle theorem and the above result are proved in a joint paper with M. Fairon, which is about to appear. I will talk about generalizations at the workshop.