Inequivalent quantizations of the three-particle Calogero model constructed by separation of variables

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Some motivations for studying Calogero type models:

– 'Alcoholics searching for keys under the lamp' (Calogero 71).
– Relations to CFT, Seiberg-Witten theory and black holes.
– Infinite particle limits relevant in condensed matter theory.
– Symmetric spaces, Lie groups, special functions.
– From the beauty of integrability to testing codes.
– Inequivalent quantizations *are* related to anomalies, defects, point interactions and *have* interesting applications.
Their study involves functional analytic aspects of Q.M.

Formally, the Calogero-Moser Hamiltonian reads

$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i=2}^{N}\sum_{j=1}^{i-1}\{\frac{1}{4}m\omega^2(x_i - x_j)^2 + g(x_i - x_j)^{-2}\}$$

Calogero assumed $g > -\frac{\hbar^2}{4m}$, presented exact solution for any N. For two particles, relative motion is governed by

$$H_y = -\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{1}{2}m\omega^2 y^2 + \frac{g}{2}y^{-2}.$$

H_y (on the minimal domain) is essentially self-adjoint only if $g \geq \frac{3\hbar^2}{4m}$, otherwise it admits ($U(2)$-family of) inequivalent self-adjoint extensions. ('Tunneling' effect, Tsutsui et al 2002.)
Energy spectrum is not bounded from below if $g < -\frac{\hbar^2}{4m}$.
Interesting inequivalent quantizations are expected *for any* N, if $-\frac{\hbar^2}{4m} \leq g < \frac{3\hbar^2}{4m}$.

Separation into spherical and radial Hamiltonians

One has $H = H_0 + H_{rel}$, where H_0 belongs to center of mass and

$$H_{rel} = H_r + r^{-2} H_\Omega$$

$$H_r = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} - \frac{\hbar^2}{2m}\frac{N-2}{r}\frac{d}{dr} + \frac{1}{4}Nm\omega^2 r^2$$

$$H_\Omega = -\frac{\hbar^2}{2m}\Delta_\Omega + g\sum_{i=2}^{N}\sum_{j=1}^{i-1}[r/(x_i - x_j)]^2$$

r: Radial variable on $\mathbf{R}^{N-1}$ spanned by the relative (Jacobi) coordinates of the particles. Δ_Ω: Standard Laplacian on S^{N-2}. Ω: collection of angle coordinates on the sphere $S^{N-2} \subset \mathbf{R}^{N-1}$.

Calogero constructed an orthogonal basis of $L^2(\mathbf{R}^{N-1})$ in the factorized form $R_{E,\lambda}(r)\eta_\lambda(\Omega)$, where

$$H_\Omega \eta_\lambda = \lambda \eta_\lambda, \qquad H_{r,\lambda} R_{E,\lambda} = E R_{E,\lambda} \quad \text{with} \quad H_{r,\lambda} = H_r + \lambda r^{-2}.$$

This is equivalent to defining self-adjoint domains for the angular and radial Hamiltonians H_Ω and $H_{r,\lambda}$.

Since $L^2(\mathbf{R}^{N-1}) = L^2(\mathbf{R}_+, r^{N-2}dr) \otimes L^2(S^{N-2})$,

$$\mathcal{H}_{r,\lambda} \equiv r^{\frac{N-2}{2}} \circ H_{r,\lambda} \circ r^{\frac{2-N}{2}} = -\frac{\hbar^2}{2m}\frac{d^2}{dr^2} + \frac{N}{4} m\omega^2 r^2 + \frac{\hbar^2}{8m}\frac{(N-2)(N-4)}{r^2} + \frac{\lambda}{r^2}$$

must be self-adjoint on $L^2(\mathbf{R}_+, dr)$.

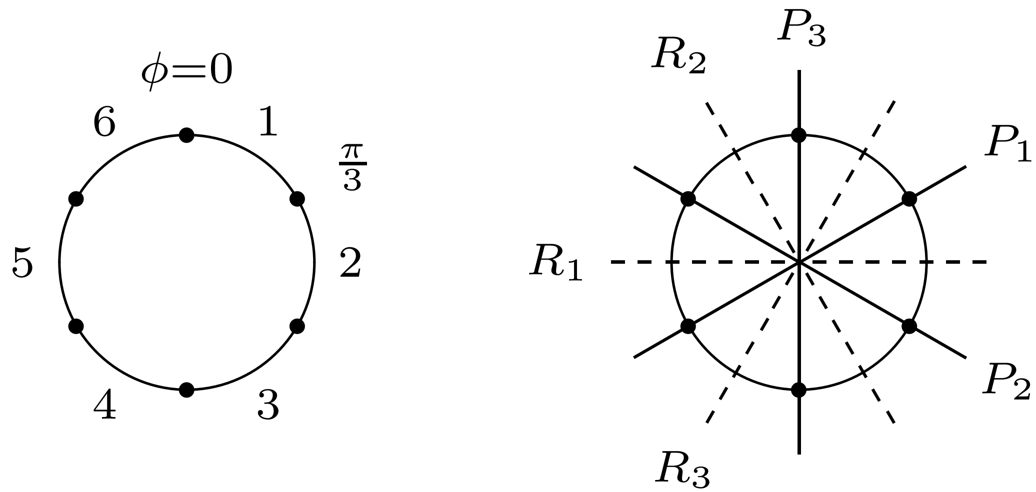
Inequivalent radial quantizations if $\frac{\hbar^2}{8m}(N-2)(N-4) + \lambda < \frac{3\hbar^2}{8m}$.

(Investigated by Basu-Mallick et al, Wipf et al 2002.)

If $\frac{\hbar^2}{8m}(N-2)(N-4) + \lambda < -\frac{\hbar^2}{8m}$, then (energy) spectrum is **not** bounded from below.

For $\mathbf{N=3}$, with $\hbar = 2m = 1$, the angular Hamiltonian becomes

$$M := H_\Omega = -\frac{d^2}{d\phi^2} + \frac{g}{2}\frac{9}{\sin^2 3\phi}.$$ Naively, M has D_6 symmetry.



The angular configuration space S^1, with the six singular points and the six 'sectors' between the consecutive singularities (left), and with the axes of the reflection symmetries of M (right).

Wish to maintain the D_6 symmetry, generated by the particle permutations and parity, in the inequivalent quantizations of M.

Character table of the dihedral group D_6

conjugacy class	$\{e\}$	$\{R_i\}$	$\{P_i\}$	$\{\mathcal{R}^{\pm 1}_{\pi/3}\}$	$\{\mathcal{R}^{\pm 2}_{\pi/3}\}$	$\{\mathcal{R}^{3}_{\pi/3}\}$
χ^{++}	1	1	1	1	1	1
χ^{-+}	1	-1	1	-1	1	-1
χ^{+-}	1	1	-1	-1	1	-1
χ^{--}	1	-1	-1	1	1	1
$\chi^{(2)}$	2	0	0	1	-1	-2
$\tilde{\chi}^{(2)}$	2	0	0	-1	-1	2

Self-adjoint versions of M

$\mathcal{D}_0 = C_0^\infty(S^1 \setminus \mathcal{S})$: minimal domain, $\mathcal{S}$: set of 6 singular points
$\mathcal{D}_1 \subset L^2(S^1)$: maximal domain for *differential operator* M
($\mathcal{D}_1 \ni \psi$: ψ, ψ' absolutely continuous, ψ, $M\psi$ square integrable)

Deficiency indices of $M_{\mathcal{D}_0}$ are (12,12) for g in our range. One has $M_{\mathcal{D}_0}^+ = M_{\mathcal{D}_1}$ and self-adjoint extensions of $M_{\mathcal{D}_0}$ are restrictions of $M_{\mathcal{D}_1}$ obtained by imposing suitable boundary conditions at the singular points $\mathcal{S}$.

Local *self-adjoint boundary conditions: ensure continuity of probability current at $\mathcal{S}$. Can be described in the form*

$$(U_\theta - \mathbf{1}_2)B_\theta(\psi) + \mathrm{i}(U_\theta + \mathbf{1}_2)B'_\theta(\psi) = 0, \qquad \forall \theta \in \mathcal{S}$$

with 'boundary values' and 'connection matrices' $\forall\, U_\theta \in U(2)$.

Let φ_1^θ, φ_2^θ be real eigenfunctions of M around $\theta \in \mathcal{S}$ normalized by the Wronskian condition $W[\varphi_1^\theta, \varphi_2^\theta] = 1$. Then the 'boundary values' $W[\psi, \varphi_k^\theta]_{\theta\pm} = \lim_{\phi\to\theta\pm 0} W[\psi, \varphi_k^\theta](\phi)$ **exist** for any $\psi \in \mathcal{D}_1$. Boundary conditions require the vanishing of some linear combinations of the boundary values.

We choose auxiliary 'reference modes' φ_k^θ ($k = 1, 2$, $i = 1, 2, 3$, $\theta \in \mathcal{S}$) as

$$\varphi_k^{R_i\theta}(\phi) = (-1)^k \varphi_k^\theta(R_i\phi) \qquad \varphi_k^0(-\phi) = (-1)^k \varphi_k^0(\phi).$$

and define the 'boundary vectors'

$$B_\theta(\psi) := \begin{bmatrix} W[\psi, \varphi_1^\theta]_{\theta+} \\ W[\psi, \varphi_1^\theta]_{\theta-} \end{bmatrix}, \quad B'_\theta(\psi) := \begin{bmatrix} W[\psi, \varphi_2^\theta]_{\theta+} \\ -W[\psi, \varphi_2^\theta]_{\theta-} \end{bmatrix}, \quad \theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}$$

$$B_\theta(\psi) := \begin{bmatrix} W[\psi, \varphi_1^\theta]_{\theta-} \\ W[\psi, \varphi_1^\theta]_{\theta+} \end{bmatrix}, \quad B'_\theta(\psi) := \begin{bmatrix} -W[\psi, \varphi_2^\theta]_{\theta-} \\ W[\psi, \varphi_2^\theta]_{\theta+} \end{bmatrix}, \quad \theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

$\forall g \in D_6$ gives unitary operator $\widehat{g}$ on $L^2(S^1)$: $(\widehat{g}\psi)(\phi) = \psi(g^{-1}(\phi))$. This is symmetry if compatible with the boundary condition, i.e., if $\widehat{g}$ preserves the domain of the self-adjoint angular Hamiltonian.

The previously described self-adjoint local boundary condition

$$(U_\theta - \mathbf{1}_2)B_\theta(\psi) + \mathrm{i}(U_\theta + \mathbf{1}_2)B'_\theta(\psi) = 0, \qquad \forall \theta \in \mathcal{S},$$

admits the D_6 symmetry iff $U_\theta = U$ constant and $\sigma_1 U \sigma_1 = U$.

$M^U \equiv M_{\mathcal{D}_U}$: self-adjoint Hamiltonian with domain $\mathcal{D}_U \subset \mathcal{D}_1$

$$U = e^{\mathrm{i}\alpha I} e^{\mathrm{i}\beta\sigma_1} = e^{\mathrm{i}\alpha} \begin{pmatrix} \cos\beta & \mathrm{i}\sin\beta \\ \mathrm{i}\sin\beta & \cos\beta \end{pmatrix} := \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{B} & \mathcal{A} \end{pmatrix}$$

$\mathcal{B} = 0$: *separating cases*, six independent sectors on S^1
$\mathcal{B} \neq 0$: *non-separating cases*, unique continuation of the wave function through the singular points
($U = -\mathbf{1}_2$: 'Dirichlet', $U = \mathbf{1}_2$ 'Neumann', $U = \sigma_1$: 'free' case)

Local eigenfunctions of $M = -\frac{d^2}{d\phi^2} + \frac{9\nu(\nu-1)}{\sin^2 3\phi}$ with $1/2 < \nu < 3/2$

$$v_{1,\mu}(\phi) := |\sin 3\phi|^{\nu} F\left(\frac{\nu-\mu}{2}, \frac{\nu+\mu}{2}, \nu+\frac{1}{2}; \sin^2 3\phi\right)$$
$$v_{2,\mu}(\phi) := |\sin 3\phi|^{1-\nu} F\left(\frac{1-\nu-\mu}{2}, \frac{1-\nu+\mu}{2}, -\nu+\frac{3}{2}; \sin^2 3\phi\right),$$

with the hypergeometric function $F(a,b,c;z)$, are eigenfunctions of eigenvalue

$$\lambda = (3\mu)^2 \quad \text{for any} \quad \mu.$$

But singular at $sin^2 3\phi = 1$ and don't satisfy boundary condition. We fix the boundary condition using the reference modes

$$\varphi_1^0(\phi) = (3(2\nu-1))^{-\frac{1}{2}} v_{1,\mu_0}(\phi)[\Theta(\phi) - \Theta(-\phi)]$$
$$\varphi_2^0(\phi) = -(3(2\nu-1))^{-\frac{1}{2}} v_{2,\mu_0}(\phi),$$

and wish to determine the spectrum of M^U.

Some auxiliary functions

To cancel the singularity of $v_{i,\mu}$ at $\frac{\pi}{6}$, we need the limiting values

$$a_i(\mu) := \lim_{\phi \to \frac{\pi}{6} - 0} v_{i,\mu}(\phi), \qquad b_i(\mu) := \lim_{\phi \to \frac{\pi}{6} - 0} \partial_\phi v_{i,\mu}(\phi)$$

Explicitly,

$$a_1(\mu) = \frac{\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\nu+1+\mu}{2})\Gamma(\frac{\nu+1-\mu}{2})}, \qquad a_2(\mu) = \frac{\Gamma(-\nu + \frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{-\nu+2+\mu}{2})\Gamma(\frac{-\nu+2-\mu}{2})}$$

$$b_1(\mu) = \frac{6\,\Gamma(\nu + \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{\nu+\mu}{2})\Gamma(\frac{\nu-\mu}{2})}, \qquad b_2(\mu) = \frac{6\,\Gamma(-\nu + \frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{-\nu+1+\mu}{2})\Gamma(\frac{-\nu+1-\mu}{2})}$$

By using these, on sector 1 we can introduce even and odd smooth eigenfunctions with respect to reflection through $\frac{\pi}{6}$.

The eigenfunctions smooth on $S^1 \setminus \mathcal{S}$, supported on sector 1 are

$$\eta^1_{+,\mu}(\phi) = \begin{cases} b_2(\mu)v_{1,\mu}(\phi) - b_1(\mu)v_{2,\mu}(\phi) & \text{if } 0 < \phi \leq \frac{\pi}{6} \text{ mod } 2\pi \\ b_2(\mu)v_{1,\mu}(\frac{\pi}{3} - \phi) - b_1(\mu)v_{2,\mu}(\frac{\pi}{3} - \phi) & \text{if } \frac{\pi}{6} \leq \phi < \frac{\pi}{3} \text{ mod } 2\pi \\ 0 & \text{otherwise} \end{cases}$$

$$\eta^1_{-,\mu}(\phi) = \begin{cases} a_2(\mu)v_{1,\mu}(\phi) - a_1(\mu)v_{2,\mu}(\phi) & \text{if } 0 < \phi \leq \frac{\pi}{6} \text{ mod } 2\pi \\ -a_2(\mu)v_{1,\mu}(\frac{\pi}{3} - \phi) + a_1(\mu)v_{2,\mu}(\frac{\pi}{3} - \phi) & \text{if } \frac{\pi}{6} \leq \phi < \frac{\pi}{3} \text{ mod } 2\pi \\ 0 & \text{otherwise} \end{cases}$$

and the ones supported on the other five sectors are

$$\eta^k_{\pm,\mu}(\phi) = \eta^1_{\pm,\mu}(\phi - (k-1)\frac{\pi}{3}), \quad \text{for} \quad k = 2, \dots, 6.$$

The most general smooth eigenfunction for any eigenvalue $9\mu^2$,

$$\eta_\mu(\phi) = \sum_{k=1}^{6} \left(C^k_+ \eta^k_{+,\mu}(\phi) + C^k_- \eta^k_{-,\mu}(\phi) \right), \qquad \forall C^k_\pm \quad \text{constants,}$$

is square integrable, but does **not always** lie in the domain $\mathcal{D}_U$.

The eigenvalues $\lambda = (3\mu)^2$ of M^U are found as the solutions of

$$F_A(\mu) := \frac{\Gamma(\frac{1+\nu+\mu}{2})\Gamma(\frac{1+\nu-\mu}{2})}{\Gamma(\frac{2-\nu+\mu}{2})\Gamma(\frac{2-\nu-\mu}{2})} = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(-\nu+\frac{3}{2})} \tan\frac{\alpha\pm\beta}{2}$$

or

$$F_B(\mu) := \frac{\Gamma(\frac{\nu+\mu}{2})\Gamma(\frac{\nu-\mu}{2})}{\Gamma(\frac{1-\nu+\mu}{2})\Gamma(\frac{1-\nu-\mu}{2})} = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(-\nu+\frac{3}{2})} \tan\frac{\alpha\pm\beta}{2}$$

or $F_2(\mu) = \pm\frac{1}{2}$ with

$$F_2(\mu) := \frac{\sin\alpha}{\sin\beta}\frac{cos\pi\mu}{\cos\pi\nu} + \frac{\cos\beta-\cos\alpha}{(6\nu-3)\sin\beta}(a_1b_1)(\mu) + \frac{\cos\beta+\cos\alpha}{(6\nu-3)\sin\beta}(a_2b_2)(\mu)$$

The corresponding eigenfunctions can be written down explicitly.
We have to consider both $\mu \in \mathbf{R}_+$ and $\mu \in \mathrm{i}\mathbf{R}_+$.

D_6 classification of the eigenstates in the separating ($\beta = 0$) case

Only the equations $F_A(\mu) = \text{constant}_A$ and $F_B(\mu) = \text{constant}_B$ arise and each eigenvalue has multiplicity 6.
The characters χ^A_μ, χ^B_μ on the respective eigensubspaces satisfy

$$\chi^A_\mu = \chi^{-+} + \chi^{--} + \chi^{(2)} + \tilde{\chi}^{(2)}, \qquad \chi^B_\mu = \chi^{++} + \chi^{+-} + \chi^{(2)} + \tilde{\chi}^{(2)}.$$

The corresponding 'bosonic' and 'fermionic' states have the form

$$\eta_\mu^{A+} = \sum_{k=1}^{6} (-1)^{k+1} \eta^k_{-,\mu}, \qquad \eta_\mu^{A-} = \sum_{k=1}^{6} \eta^k_{-,\mu},$$

$$\eta_\mu^{B+} = \sum_{k=1}^{6} \eta^k_{+,\mu}, \qquad \eta_\mu^{B-} = \sum_{k=1}^{6} (-1)^{k+1} \eta^k_{+,\mu}.$$

'Type 2' sates are associated with the characters $\chi^{(2)}$ and $\tilde{\chi}^{(2)}$.

D_6 classification of the eigenstates in the non-separating case

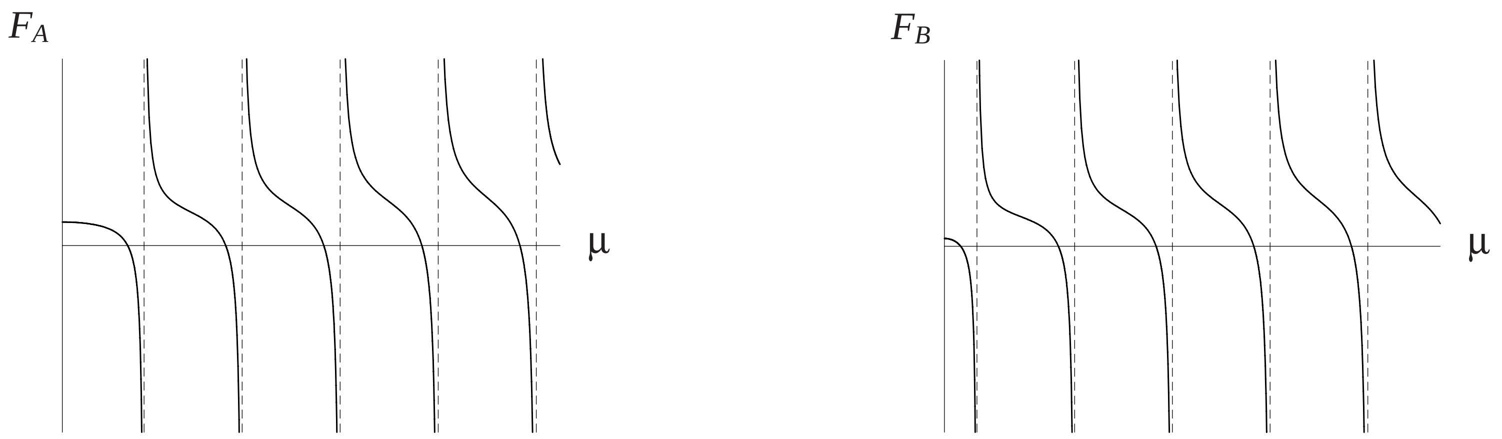
The eigenvalues arising from the solutions of

$$F_A(\mu) = \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(-\nu + \frac{3}{2})} \tan \frac{\alpha \pm \beta}{2}, \quad F_B(\mu) = \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(-\nu + \frac{3}{2})} \tan \frac{\alpha \pm \beta}{2}$$

have multiplicity 1. The corresponding eigenstate belongs to the 'type 1' (dimension 1) representation with character $\chi^{-\pm}$ in case A and $\chi^{+\pm}$ in case B, respectively. These states have the same form as in the separating case.

The eigenvalues arising from $F_2(\mu) = \pm\frac{1}{2}$ have multiplicity 2. The corresponding 'type 2' (dim. 2) irrep. of D_6 has character $\chi^{(2)}$ for $\frac{1}{2}$ and character $\tilde{\chi}^{(2)}$ for $-\frac{1}{2}$ on the right hand side.

The shape of the functions F_A and F_B



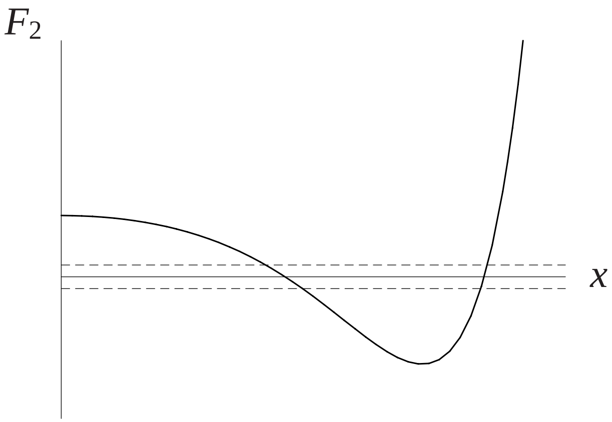
F_A and F_B as the function of μ for $\mu \geq 0$, with $\nu = 2/3$.
The function F_A (resp. F_B) diverges at μ_m^∞ (resp. at $\bar{\mu}_m^\infty$):

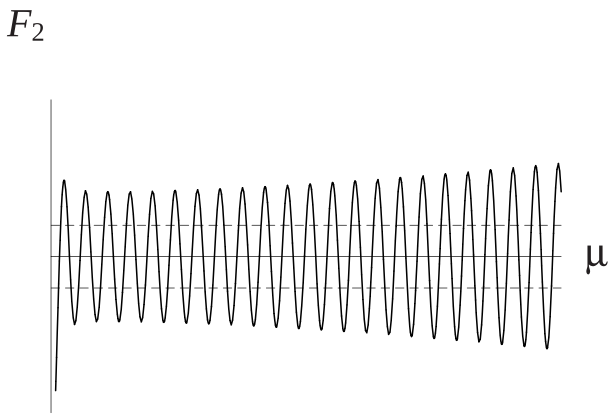
$$\mu_m^\infty = (\nu + 1) + 2m, \quad \bar{\mu}_m^\infty = \nu + 2m \quad (m = 0, 1, 2, \ldots)$$

The function F_A (resp. F_B) vanishes at μ_m^0 (resp. at $\bar{\mu}_m^0$):

$$\mu_m^0 = (2 - \nu) + 2m, \quad \bar{\mu}_0^0 = |1 - \nu|, \quad \bar{\mu}_{m+1}^0 = (3 - \nu) + 2m.$$

One can prove that $F_A(\mathrm{i}x)$ and $F_B(\mathrm{i}x)$ increase monotonically from a positive value to $+\infty$ as x runs from 0 to ∞. Thus $F_A(\mu) = constant$ and $F_B(\mu) = constant$ can lead to at most one negative eigenvalue, $\lambda = 9\mu^2$. Positive eigenvalues can be obtained explicitly if $constant \in \{0, \pm\infty\}$. As for the 'type 2' case,





An example of $F_2(\mu)$, for imaginary values of $\mu = \mathrm{i}x$ (left) and for real values of μ (right). The dashed lines lie at $\pm\frac{1}{2}$.

Explicitly finds 'type 2' eigenvalues if last two terms of F_2 vanish

$$F_2(\mu) := \frac{\sin\alpha}{\sin\beta}\frac{cos\pi\mu}{\cos\pi\nu} + \frac{\cos\beta - \cos\alpha}{(6\nu-3)\sin\beta}(a_1b_1)(\mu) + \frac{\cos\beta + \cos\alpha}{(6\nu-3)\sin\beta}(a_2b_2)(\mu)$$

This happens iff $U = \pm\sigma_1$. Otherwise, qualitative analysis using

$$(a_1b_1)(\mu) = \frac{6\,\Gamma^2(\nu+\frac{1}{2})2^{2(\nu-1)}}{\Gamma(\nu+\mu)\Gamma(\nu-\mu)},\ \ (a_2b_2)(\mu) = \frac{6\,\Gamma^2(-\nu+\frac{3}{2})2^{-2\nu}}{\Gamma(1-\nu+\mu)\Gamma(1-\nu-\mu)}$$

'Stability result' on negative eigenvalues of M^U:

Suppose that all eigenvalues of M^U are positive for $U = U(\alpha_0, \beta_0)$ and (α_0, β_0) are generic in the sense that

$$|\tan\frac{\alpha_0 \pm \beta_0}{2}| < \infty, \quad \sin\beta_0 \neq 0, \quad (\cos\alpha_0 + \cos\beta_0) \neq 0.$$

Then M^U *with* $U(\alpha,\beta)$ *has only positive eigenvalues for any* (α,β) *near to* (α_0,β_0)*.* The property of admitting a negative eigenvalue is also stable generically under small perturbations of U.

The radial Hamiltonian: $\mathcal{H}_{r,\lambda} = -\frac{d^2}{dr^2} + \frac{3}{8}\omega^2 r^2 + \frac{\lambda - \frac{1}{4}}{r^2}$ must be self-adjoint on a domain in $L^2(\mathbf{R}_+, dr)$. This domain is unique if $\lambda \geq 1$, and then obtains the eigenvalues/eigenfunctions

$$E_{m,\lambda} = 2c(2m + 1 + \sqrt{\lambda}), \quad c := \sqrt{\frac{3}{8}}\omega, \quad m = 0, 1, 2, \ldots,$$

$$\rho_{m,\lambda}(r) = r^{\frac{1}{2}+\sqrt{\lambda}} e^{-\frac{1}{2}cr^2} L_m^{\sqrt{\lambda}}(cr^2),$$

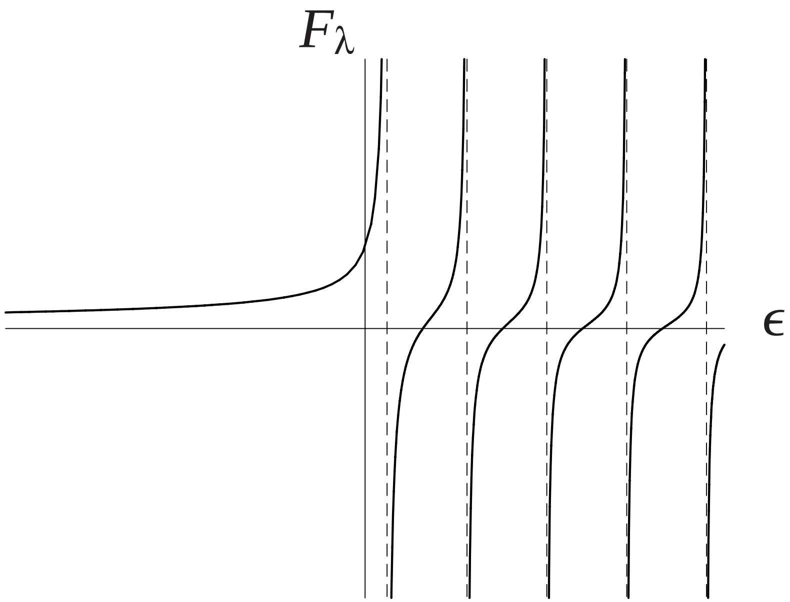
where $L_m^{\sqrt{\lambda}}$ is the Laguerre polynomial (exercise in Landau Q.M.).

There is a one-parameter family of radial quantizations if $\lambda < 1$. We reproduce results of (Wipf et al, Basu-Mallick et al, 2002) for $0 < \lambda < 1$, with independent proofs, and also prove that **energy is not bounded from below if** $\lambda < 0$.

The energy spectrum for $0 < \lambda < 1$ is found as the solution of

$$F_\lambda(\epsilon) := \frac{\Gamma(-\epsilon + \frac{1-\sqrt{\lambda}}{2})}{\Gamma(-\epsilon + \frac{1+\sqrt{\lambda}}{2})} = -\frac{\Gamma(-\sqrt{\lambda})}{\Gamma(\sqrt{\lambda})}\kappa(\lambda) \quad \text{with} \quad \epsilon := \frac{E}{4c}$$

and 'quantization parameter' $\kappa \in \mathbf{R} \cup \{\infty\}$. The shape of F_λ is



$F_\lambda = 0$ at $\frac{1+\sqrt{\lambda}}{2} + m$ and $|F_\lambda| = \infty$ at $\frac{1-\sqrt{\lambda}}{2} + m$ for any $m \in \mathbf{Z}_+$.

Energy spectrum in the four explicitly solvable cases:

'Dirichlet' case, $U = -\mathbf{1}_2$ (reproduces Calogero's result):

$$E^A_{mn} = 2c\,(2m + 1 + 3(2n + 1 + \nu))\,, \;\; E^B_{mn} = 2c\,(2m + 1 + 3(2n + \nu))$$

'Neumann' case, $U = \mathbf{1}_2$:

$$E^A_{mn} = 2c\,(2m + 1 + 3(2n + 1 + (1 - \nu)))\,,$$

$$E^B_{mn} = 2c\,(2m + 1 + 3|2n + (1 - \nu)|)\,, \qquad m, n = 0, 1, 2, \ldots .$$

Two new isospectral cases, $U = \pm\sigma_1$: Union of the eigenvalues in the Dirichlet and Neumann cases (for type 1 states), together with the type 2 eigenvalues, with $\Delta(\nu) := \frac{1}{\pi} \arccos\left(\frac{1}{2} \cos \pi\nu\right)$,

$$\begin{aligned}
E^{(2)+}_{mn} &= 2c\,(2m + 1 + 3(2n + (1 - \Delta(\nu))))\,, \\
\tilde{E}^{(2)+}_{mn} &= 2c\,(2m + 1 + 3(2n + 1 + (1 - \Delta(\nu)))\,, \\
\tilde{E}^{(2)-}_{mn} &= 2c\,(2m + 1 + 3(2n + \Delta(\nu)))\,, \\
E^{(2)-}_{mn} &= 2c\,(2m + 1 + 3(2n + 1 + \Delta(\nu)))\,.
\end{aligned}$$

Concluding remarks

1. System with $U = \sigma_1$ 'free' boundary condition tends to two-dimensional harmonic oscillator as coupling constant $g \to 0$.
2. What about generalization to arbitrary particle number N?
3. Does integrability select boundary conditions? What about the scattering problem?
4. Separation of variables in different coordinate systems may lead to widely different quantizations. Illustration using the $N = 2$ Calogero model.
5. By the techniques developed, can study, e.g., the $SU(1,1)$ anomaly for the radial equation, or the 2-particle dihedral D_n type Calogero models.
6. Deficiency indices of the original (non-separated) Hamiltonian are probably infinite, how to investigate it?