Trigonometric real form of the spin RS model of Krichever and Zabrodin

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We study a generalization of the famous integrable systems exemplified by the trigonometric Sutherland model

$$H_{\text{trigo}-\text{Suth}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{\sin^2(q_k - q_j)}$$

and the trigonometric Ruijsenaars–Schneider (RS) model

$$H_{\text{trigo}-\text{RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j\neq k} \left[1 + \frac{x^2}{\sin^2(q_k - q_j)}\right]^{\frac{1}{2}}$$

These models describe integrable interactions of n points moving on the circle, and generalize the rational Calogero–Moser model having the Hamiltonian

$$H_{\text{CM}} = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{(q_k - q_j)^2}$$

A powerful approach to these systems consists in presenting them as Hamiltonian reductions of 'obviously integrable' simple systems on suitable higher dimensional phase spaces.

For example, to derive the Calogero model (OP [76], KKS [78]) consider the phase space $T^*\mathrm{i}u(n) \simeq \mathrm{i}\mathfrak{u}(n) \times \mathrm{i}\mathfrak{u}(n) := \{(Q, P)\}$ equipped with the commuting family of 'free' Hamiltonians $\{\mathrm{tr}(P^k)\}$. Reduce by the 'conjugation action' of $\mathrm{U}(n)$ using the moment map constraint

$$[Q, P] = \mathrm{i}x \sum_{j \neq k} E_{j,k}$$

A model of the reduced phase space is defined by the 'gauge slice' whose elements (Q, P) have the form

$$Q = \mathrm{diag}(q_1, \dots, q_n), \qquad q_1 > \cdots > q_n,$$

and

$$P = \mathrm{diag}(p_1, \dots, p_n) + \mathrm{i}x \sum_{j \neq k} \frac{E_{jk}}{q_j - q_k}.$$

Then $\mathrm{tr}(dP \wedge dQ) = \sum_{k=1}^{n} dp_k \wedge dq_k$ gives the reduced symplectic form and $\frac{1}{2}\mathrm{tr}(P^2)$ yields the rational Calogero Hamiltonian. The family $\{\mathrm{tr}(P^k)\}$ guarantees its Liouville integrability.

To sketch another example, let us start with the symplectic manifold

$$T^*\mathsf{U}(n) \times \mathbb{C}^{n\times d}$$

for some natural number d. The second factor encodes nd copies of the symplectic vector space $\mathbb{R}^2$. Denote the general element of $\mathbb{C}^{n\times d}$ as the matrix S_{aj}, and let (g, J) stand for the general element of the cotangent bundle, trivialized by right-translations. The conventions are such that the following formula gives a Poisson map into $\mathfrak{u}(n)$, identified with its own dual space:

$$\Phi(g, J, S) = J - g^{-1}Jg + \mathrm{i}SS^\dagger$$

This is the moment map for the following Hamiltonian action of $\mathsf{U}(n)$:

$$A_\eta : (g, J, S) \mapsto (\eta g \eta^{-1}, \eta J \eta^{-1}, \eta S), \quad \forall \eta \in U(n).$$

We reduce by imposing the moment map constraint

$$\Phi(g, J, S) = \mathrm{i}c\mathbf{1}_n,$$

where c is a non-zero real, positive constant. Now, on a dense open part we can go to a partial gauge fixing, where $g = \exp(\mathrm{i}q)$ with q being a real diagonal matrix having different eigenvalues $q_1, q_2, \dots, q_n$, so that $e^{\mathrm{i}q} \in \mathbb{T}^n_{\text{reg}}$. Then we get

$$J_{ab} = \mathrm{i}p_a\delta_{ab} - \mathrm{i}(1-\delta_{ab})\frac{(S_a, S_b)}{1-\exp(\mathrm{i}(q_b - q_a))}$$

with arbitrary real p_a. Here, $(S_a, S_b) := \sum_{j=1}^d S_{aj}\bar{S}_{bj}$ The reduced 'free' Hamiltonian reads

$$H = -\frac{1}{2}\mathrm{tr}(J^2) = \frac{1}{2}\sum_{a=1}^n p_a^2 + \frac{1}{4}\sum_{a\neq b}\frac{|(S_a, S_b)|^2}{\sin^2\frac{q_a - q_b}{2}}.$$

The row-vector $S_a := [S_{a1}, \dots, S_{ad}]$ is interpreted as some internal, 'spin' degree of freedom attached to the particle with coordinate q_a. The moment map constraint becomes equivalent to $(S_a, S_a) = c$, i.e., S_a is a non-zero $\mathbb{C}^d$-vector of fixed length. Therefore the residual gauge transformations, which are given by the torus $\mathbb{T}^n$ and by the permutation group S_n, act freely. We obtain the reduced phase space

$$\left(T^*\mathbb{T}^n_{\text{reg}} \times (\mathbb{CP}^{d-1} \times \cdots \times \mathbb{CP}^{d-1})\right)/S_n$$

with n-copies of the complex projective space, if $d > 1$. If $d = 1$, then we get the spinless Sutherland model. If $d > 1$ this the 'spin Sutherland model' due to Gibbons and Hermsen.

The model of our present interest was introduced by Krichever and Zabrodin in 1995. It deals with the dynamics of 'particle positions' x_i $(i = 1, \ldots, n)$ and d-component, complex row vectors c_i, and column vectors a_i. The 'individual spins' enter the 'composite spin variables' $F_{ij} := c_i \cdot a_j := \sum_{\alpha=1}^{d} c_i^\alpha a_j^\alpha$, and the equations of motion read

$$\dot{x}_i = F_{ii}, \quad \dot{a}_i^\alpha = \lambda_i a_i^\alpha + \sum_{k \neq i} V(x_{ik}) a_k^\alpha F_{ki}, \quad \dot{c}_j^\alpha = -\lambda_j c_j^\alpha - \sum_{k \neq j} V(x_{kj}) c_k^\alpha F_{jk}$$

where $x_{ik} := x_i - x_k$. In the elliptic case the 'potential' is $V(x) = \zeta(x) - \zeta(x + \gamma)$ with the Weierstrass zeta-function and an arbitrary 'coupling constant' $\gamma \neq 0$. These imply the second order equations

$$\ddot{x}_i = \sum_{j \neq i} F_{ij} F_{ji} \left[V\left(x_{ij}\right) - V\left(x_{ji}\right) \right].$$

The parameters λ_i are arbitrary, and the 'physical observables' are invariant with respect to arbitrary rescalings $a_i \mapsto \Lambda_i^{-1} a_i$, $c_i \mapsto \Lambda_i c_i$.

Krichever and Zabrodin derived these equations from the dynamics of the poles of the elliptic solutions of the 2D non-Abelian Toda lattice, and asked about their Hamiltonian interpretation and integrability.

In the rational case, $V^{\mathrm{rat}}(x) = x^{-1} - (x+\gamma)^{-1}$, the answers were provided by Arutyunov and Frolov (1998), who re-derived the model via Hamiltonian reduction of a spin extension of the cotangent bundle of $\mathrm{GL}(n,\mathbb{C})$, $T^*\mathrm{GL}(n,\mathbb{C}) \times \mathbb{C}^{2nd}$. Twenty years later, the trigonometric/hyperbolic case was treated, first by Chalykh and Fairon and then by Arutyunov and Olivucci, applying quasi-Hamiltonian reduction and Hamiltonian reduction techniques, respectively. (The two methods led to different Hamiltonian structures for the model.)

Krichever (1998) proved the existence of Hamiltonian structure in the general case.

The pioneering papers on Calogero and Ruijsenaars type system were devoted to point particles moving along the **real** line or circle. However, all the above mentioned works deal with holomorphic systems. The real forms require separate attention, which poses open problems.

In this talk, we inquire about the trigonometric real form defined by taking $V(x) := \cot(x) - \cot(x - \mathrm{i}\gamma)$ with a real, positive γ, and setting $x_j := \frac{1}{2} q_j$ where the q_j are real and are regarded as angles, and also setting $c_i^\alpha = (a_i^\alpha)^* =: v(\alpha)_i$, i.e., $c = a^\dagger$. In this case the second order equations of motion read

$$\frac{1}{2}\ddot{q}_i = \sum_{j \neq i} F_{ij} F_{ji} \frac{2\cot(\frac{q_{ij}}{2})}{1 + \sinh^{-2}(\gamma)\,\sin^2(\frac{q_{ij}}{2})}.$$

If $d = 1$, then $F_{ij} F_{ji} = |F_{ij}|^2 = F_{ii} F_{jj}$ and the gauge invariant content of the model is governed by the chiral RS Hamiltonian

$$\mathcal{H}^+_{\mathrm{RS}} = \sum_i e^{2p_i} \prod_{j \neq i} \left[1 + \frac{\sinh^2 \gamma}{1 + \sin^2 \frac{q_i - q_j}{2}} \right]^{\frac{1}{2}}$$

via the change of variables $F_{jj} = |v_i|^2 = e^{2p_j} \prod_{i \neq j} \left[1 + \frac{\sinh^2 \gamma}{1 + \sin^2 \frac{q_i - q_j}{2}} \right]^{\frac{1}{2}}$.

The trigonometric RS model was derived by L.F. and Klimčík (2009) by Hamiltonian reduction of the Heisenberg double of the Poisson–Lie group $\mathrm{U}(n)$, which served as the starting point for the reported work.

The rest of the talk

- The ‘free’ system to be reduced

- The moment map and the definition of the reduction

- The first model of the reduced phase space and connection with the Gibbons–Hermsen model

- The reduced equations of motion and the second model of the reduced phase space

- Degenerate integrability

- Conclusion

We shall apply symplectic reduction to an 'obviously integrable' Hamiltonian system on the real, $2n^2 + 2nd$ dimensional symplectic manifold

$$\mathcal{M} = \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^{n\times d}.$$

We begin by decomposing the real Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ as $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n) + \mathfrak{b}(n)$, where $\mathfrak{b}(n)$ denotes the Lie algebra of upper triangular complex matrices having real diagonal entries. Then $\mathfrak{b}(n)$ and $\mathfrak{u}(n)$ are isotropic subalgebras with respect to the non-degenerate bilinear form

$$\langle X, Y\rangle := \Im \mathrm{tr}(XY), \quad \forall X, Y \in \mathfrak{gl}(n,\mathbb{C}).$$

Using the r-matrix on $\mathfrak{gl}(n,\mathbb{C})$ given by $R := \frac{1}{2}\Big(P_{\mathfrak{u}(n)} - P_{\mathfrak{b}(n)}\Big)$, we introduce two Poisson structures on $C^\infty(\mathrm{GL}(n,\mathbb{C}),\mathbb{R})$:

$$\{f, h\}_\pm := \langle \nabla f, R\nabla h\rangle \pm \langle \nabla' f, R\nabla' h\rangle,$$

where $\langle \nabla f(K), X\rangle := \frac{d}{dt}\Big|_{t=0} f(e^{tX}K)$ $\forall X \in \mathfrak{gl}(n,\mathbb{C})$ and $K \in \mathrm{GL}(n,\mathbb{C})$, and similar for the right-derivative $\nabla' f$.

The minus bracket makes $\mathrm{GL}(n,\mathbb{C})$ into a real Poisson–Lie group, while the plus one gives a symplectic structure. The former is called the Drinfeld double Poisson bracket and the latter the Heisenberg double Poisson bracket. $\mathrm{U}(n)$ and $\mathrm{B}(n)$ are Poisson submanifolds w.r.t. the minus bracket, and thus become Poisson–Lie groups, equipped with the inherited Poisson structures denoted $\{\ ,\ \}_U$ and $\{\ ,\ \}_B$.

The Heisenberg double goes back to Semenov-Tian-Shansky (1985), and its symplectic form was found by Alekseev and Malkin (1994). For any element $K \in \mathrm{GL}(n,\mathbb{C})$, use the Iwasawa decompositions

$$K = b_L g_R^{-1} = g_L b_R^{-1} \quad \text{with} \quad b_L, b_R \in \mathrm{B}(n),\, g_L, g_R \in \mathrm{U}(n),$$

and define the maps Λ_L, Λ_R into $\mathrm{B}(n)$ and Ξ_L, Ξ_R into $\mathrm{U}(n)$ by

$$\Lambda_L(K) := b_L, \quad \Lambda_R(K) := b_R, \quad \Xi_L(K) := g_L, \quad \Xi_R(K) := g_R.$$

Then $\quad \Omega_{GL} = \frac{1}{2}\Im \mathrm{tr}(d\Lambda_L \Lambda_L^{-1} \wedge d\Xi_L \Xi_L^{-1}) + \frac{1}{2}\Im \mathrm{tr}(d\Lambda_R \Lambda_R^{-1} \wedge d\Xi_R \Xi_R^{-1}).$

To build $(\mathcal{M}, \Omega_{\mathcal{M}})$, equip $\mathbb{C}^n = \mathbb{R}^{2n}$ with the $\mathrm{U}(n)$ covariant Poisson structure:

$$\{w_i, w_l\} = \mathrm{i}\,\mathrm{sgn}(i-l) w_i w_l, \qquad \forall 1 \leq i, l \leq n,$$
$$\{w_i, \overline{w}_l\} = \mathrm{i}\,\delta_{il}(2 + |w|^2) + \mathrm{i} w_i \overline{w}_l + \mathrm{i}\,\delta_{il} \sum_{r=1}^{n} \mathrm{sgn}(r-i)|w_r|^2 .$$

This is due to Zakrzewski (1996), and we found its symplectic form

$$\Omega_{\mathbb{C}^n} = \frac{\mathrm{i}}{2} \sum_{k=1}^{n} \frac{1}{\mathcal{G}_k} dw_k \wedge d\overline{w}_k + \frac{\mathrm{i}}{4} \sum_{k=1}^{n-1} \frac{1}{\mathcal{G}_k \mathcal{G}_{k+1}} d\mathcal{G}_{k+1} \wedge (\overline{w}_k dw_k - w_k d\overline{w}_k)$$

where $\mathcal{G}_j = 1 + \sum_{k=j}^{n} |w_k|^2$ $(j = 1, \ldots, n)$ and $\mathcal{G}_{n+1} := 1$.

We take $d > 1$ independent, $\mathbb{C}^n$-valued variables, $w^1, \dots, w^d$, called primary spins, which give $W := (w^1, \dots, w^d) \in \mathbb{C}^{n\times d}$. The so obtained Poisson bracket and symplectic form are denoted $\{\ ,\}_{\mathcal{W}}$ and $\Omega_{\mathcal{W}}$.

The phase space to be reduced is $\mathcal{M} := \mathrm{GL}(n,\mathbb{C}) \times \mathbb{C}^{n\times d}$ endowed with the symplectic form $\Omega_{\mathcal{M}} = \Omega_{GL} + \Omega_{\mathcal{W}}$ and the corresponding product Poisson structure $\{\ ,\ \}_{\mathcal{M}}$. $\mathcal{M}$ carries the Abelian Poisson algebra $\mathfrak{H}$ generated by the 'free' Hamiltonians H_k:

$$H_k(K,W) := \frac{1}{2k}\mathrm{tr}(L^k) \quad \text{with} \quad L := b_R b_R^\dagger = (K^\dagger K)^{-1}, \quad k = 1, \dots, n.$$

Along the Hamiltonian flow of H_k, we have

$$g_R(t) = \exp(\mathrm{i}L(0)^k t) g_R(0),$$

while b_R, b_L and W do not change. Therefore the arbitrary functions of b_L, b_R, W form the Poisson algebra $\mathfrak{C}$ of constants of motion, i.e., the commutant of $\mathfrak{H}$ in $C^\infty(\mathcal{M})$.

The functional dimensions of $\mathfrak{H}$ and $\mathfrak{C}$ add up to the dimension of $\mathcal{M}$ (since $b_R b_R^\dagger = g_R \left(b_L^{-1}(b_L^{-1})^\dagger\right) g_R^{-1}$). This means that the 'free' Hamiltonians $\mathfrak{H}$ form a degenerate integrable system on $\mathcal{M}$. The generic level surfaces of $\mathfrak{C}$ are n-dimensional tori. (Liouville integrability holds, too.)

Suppose that we have a Poisson map, Λ, from a symplectic manifold $\mathcal{M}$ into the Poisson–Lie group $(\mathrm{B}(n), \{\ ,\ \}_B)$. Then, for any $X \in \mathfrak{u}(n)$ the following formula defines a vector field $X_{\mathcal{M}}$ on $\mathcal{M}$:

$$\mathcal{L}_{X_{\mathcal{M}}}(\mathcal{F}) \equiv X_{\mathcal{M}}[\mathcal{F}] := \left\langle X, \{\mathcal{F}, \Lambda\}_{\mathcal{M}} \Lambda^{-1} \right\rangle, \qquad \forall \mathcal{F} \in C^{\infty}(\mathcal{M}).$$

This generates an infinitesimal left action of $\mathrm{U}(n)$. If it integrates to a global action of $\mathrm{U}(n)$, then the resulting action is Poisson, i.e., the action map $\mathcal{A} : \mathrm{U}(n) \times \mathcal{M} \to \mathcal{M}$ is Poisson. Then Λ is called the (Poisson–Lie) **moment map** for the corresponding Poisson action.

Picking a moment map value, μ, one obtains the reduced phase space

$$\mathcal{M}_{\mathrm{red}} := \Lambda^{-1}(\mu)/\mathrm{U}(n)_{\mu}$$

where $\mathrm{U}(n)_{\mu}$ is the isotropy group of μ w.r.t. dressing action of $\mathrm{U}(n)$ on $\mathrm{B}(n)$, given by $\mathrm{Dress}_g(\mu) := \Lambda_L(g\mu)$. If the action of $\mathrm{U}(n)_{\mu}$ is free, then $\mathcal{M}_{\mathrm{red}}$ is a smooth symplectic manifold. Letting $\iota_{\mu} : \Lambda^{-1}(\mu) \to \mathcal{M}$ and $\pi_{\mu} : \Lambda^{-1}(\mu) \to \mathcal{M}_{\mathrm{red}}$ denote the natural maps, one has

$$\pi_{\mu}^{*}\Omega_{\mathrm{red}} = \iota_{\mu}^{*}\Omega_{\mathcal{M}}, \ \ \pi_{\mu}^{*}\mathcal{F}_{\mathrm{red}} = \iota_{\mu}^{*}\mathcal{F}, \ \ \{\mathcal{F}, \mathcal{H}\}_{\mathcal{M}} \circ \iota_{\mu} = \{\mathcal{F}_{\mathrm{red}}, \mathcal{H}_{\mathrm{red}}\}_{\mathrm{red}} \circ \pi_{\mu}$$

for $\mathrm{U}(n)$-invariant functions on $\mathcal{M}$, with reduced symplectic form Ω_{red} and corresponding Poisson structure. (This generalization of Marsden–Weinstein reduction is due to J.–H. Lu (1990).)

We have a Poisson map $\mathbf{b}$ from $(\mathbb{C}^n, \Omega_{\mathbb{C}^n})$ to $\mathrm{B}(n)$ that satisfies

$$\mathbf{1}_n + ww^\dagger =: \mathbf{b}(w)\mathbf{b}(w)^\dagger$$

and generates the natural left action of $\mathrm{U}(n)$ on $\mathbb{C}^n$. Explicitly,

$$\mathbf{b}_{jj}(w) = \sqrt{\mathcal{G}_j/\mathcal{G}_{j+1}}, \qquad \mathbf{b}_{ij}(w) = \frac{w_i \overline{w}_j}{\sqrt{\mathcal{G}_j \mathcal{G}_{j+1}}}, \qquad \forall 1 \leq i < j \leq n.$$

Our construction is based on the product moment map $\Lambda : \mathcal{M} \to \mathrm{B}(n)$:

$$\Lambda(K, W) := \Lambda_L(K)\Lambda_R(K)\mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d),$$

for $(K, W) \in \mathcal{M} \equiv \mathrm{GL}(n, \mathbb{C}) \times \mathbb{C}^{n \times d}$.(Recall: $\Lambda_L(K) = b_L$ for $K = b_L g_R^{-1}$). We choose the moment map value $e^\gamma \mathbf{1}_n$ with an arbitrary $\gamma > 0$. The reduced phase space

$$\mathcal{M}_{\mathrm{red}} = \Lambda^{-1}(e^\gamma \mathbf{1}_n)/\mathrm{U}(n)$$

is then a smooth, and actually real analytic, symplectic manifold.

The $\mathrm{U}(n)$ action generated by Λ has a complicated form, but it simplifies in terms of suitable variables. Instead of $K, w^1, \dots, w^d$ we introduce the new variables

$$g_R, b_R \quad \text{and} \quad v(\alpha) := b_R \mathbf{b}(w^1) \cdots \mathbf{b}(w^{\alpha-1}) w^\alpha \quad \text{for} \quad 1 \leq \alpha \leq d.$$

The map $(K, w^1, \dots, w^d) \mapsto (g_R, b_R, v(1), \dots, v(d))$ is a diffeomorphism. Instead of b_R, we may use $L := b_R b_R^\dagger$. The $v(\alpha)$ are called 'dressed spins'. The Poisson–Lie action of $\mathrm{U}(n)$ is orbit-equivalent to the 'obvious action', where $\eta \in \mathrm{U}(n)$ acts as

$$(g_R, b_R, v(1), \dots, v(d)) \mapsto \left(\eta g_R \eta^{-1}, \mathrm{Dress}_\eta(b_R), \eta v(1), \dots, \eta v(d)\right)$$

and the transformation of b_R is equivalent to $L \mapsto \eta L \eta^{-1}$.

Consider the following $\mathrm{U}(n)$ invariant complex functions on $\mathcal{M}$:

$$I^k_{\alpha\beta} := \mathrm{tr}\left(v(\alpha) v(\beta)^\dagger L^k\right) = v(\beta)^\dagger L^k v(\alpha)\,, \quad 1 \leq \alpha, \beta \leq d,\ k \geq 0.$$

They belong to the commutant of the 'free' Hamiltonians, and their real and imaginary parts generate a polynomial Poisson algebra. This descends to $\mathcal{M}_{\mathrm{red}}$ and underlies the degenerate integrability of $\mathfrak{H}_{\mathrm{red}}$.

We now construct our first model of the reduced phase space. Since g_R can be diagonalized, every gauge orbit has representatives in

$$\mathcal{M}_0 := \{(Q, b_R, W) \in \Lambda^{-1}(e^\gamma \mathbf{1}_n) \mid Q \in \mathbb{T}^n\}.$$

We focus on the dense subset $\mathcal{M}_0^{\text{reg}}$ where Q is regular. Using the (diagonal $\times$ strictly upper-triangular) decomposition $\mathrm{B}(n) = \mathrm{B}(n)_0 \mathrm{B}_+(n)$, we can write

$$b_R = b_0 b_+ \quad \text{and} \quad \mathbf{b}(w^1)\mathbf{b}(w^2)\cdots\mathbf{b}(w^d) =: S(W) =: S_0(W)S_+(W).$$

Then the moment map constraint becomes equivalent to

$$S_0(W) = e^\gamma \mathbf{1}_n \quad \text{and} \quad b_+ S_+(W) = Q^{-1} b_+ Q.$$

The first equation constraints W only, while the second one permits us to express b_+ in terms of $Q = e^{\mathrm{i}q} \in \mathbb{T}^n_{\text{reg}}$ and W. The explicit formula of $b_+(Q, W)$ is given in our paper.

Note that $Q \in \mathbb{T}^n_{\text{reg}}$ and $b_0 \equiv \exp(p)$ ($p = \text{diag}(p_1, \ldots, p_n)$) are arbitrary. The reduced phase space can be parametrized Q, p and the constrained primary spins, W, up to residual gauge transformations.

In case one wish to see $b_+(Q,W)$ explicitly, define $\mathcal{I}^{a,a+j} = \frac{1}{Q_{a+j}Q_a^{-1}-1}$.
In fact, the solution of the constraint has the form $b_+^{a,a+1} = \mathcal{I}^{a,a+1}S_+^{a,a+1}$, and for $k = 2,\dots,n-a$

$b_+^{a,a+k} = \mathcal{I}^{a,a+k}S_+^{a,a+k}$ + higher order polynomials in S_+ and the $\mathcal{I}^{a,a+j}$.

In full details,

$$b_+^{a,a+k} = \sum_{\substack{m=2,\dots,k \\ (i_1,\dots,i_m)\in\mathbb{N}^m \\ i_1+\cdots+i_m=k}} \prod_{\alpha=1}^{m} \mathcal{I}^{a,a+i_1+\cdots+i_\alpha} S_+^{a+i_1+\cdots+i_{\alpha-1},a+i_1+\cdots+i_\alpha}$$

The message is that in terms of this model of the reduced phase space the symplectic form is simple, but the commuting Hamiltonians and the equations of motion are rather complicated.

The map $\phi : \mathbb{C}^{n\times d} \to \mathfrak{b}(n)_0 \simeq \mathbb{R}^n$ defined by writing $S_0(W) := \exp(\phi(W))$ is the moment map for a Hamiltonian torus action. Explicitly, this action is given by $\tau \cdot (w^1, \dots, w^d) = (\tau w^1, \dots, \tau w^d)$, $\forall \tau \in \mathbb{T}^n$.

Lemma. The moment map $\phi : \mathbb{C}^{n\times d} \to \mathfrak{b}(n)_0$ is proper, and the reduced space of primary spins, $\mathbb{C}^{n\times d}_{\mathrm{red}} := \phi^{-1}(\gamma \mathbf{1}_n)/\mathbb{T}^n$, is a smooth, compact and connected symplectic manifold of dimension $2n(d-1)$.

With the normalizer $\mathcal{N}(n)$ of $\mathbb{T}^n$, consider the regular part of $\mathcal{M}_{\mathrm{red}}$:

$$\mathcal{M}^{\mathrm{reg}}_{\mathrm{red}} = \mathcal{M}^{\mathrm{reg}}_0/\mathcal{N}(n) = (\mathcal{M}^{\mathrm{reg}}_0/\mathbb{T}^n)/S_n, \quad (S_n = \mathcal{N}(n)/\mathbb{T}^n).$$

Theorem 1. The covering space $\mathcal{M}^{\mathrm{reg}}_0/\mathbb{T}^n$ of the regular part of the reduced phase space can be identified with the symplectic manifold

$$T^*\mathbb{T}^n_{\mathrm{reg}} \times \mathbb{C}^{n\times d}_{\mathrm{red}}$$

equipped with its natural product symplectic structure. The dense open submanifold $\mathcal{M}^{\mathrm{reg}}_{\mathrm{red}} \subseteq \mathcal{M}_{\mathrm{red}}$ is connected, and consequently $\mathcal{M}_{\mathrm{red}}$ is also connected.

Let us connect our reduced system with the spin Sutherland model of Gibbons and Hermsen (1984). For this, we introduce a positive 'scaling parameter' ϵ and make the replacements

$$p \to \epsilon p, \quad W \to \epsilon^{\frac{1}{2}} W, \quad Q \to Q, \quad \Omega_{\mathcal{M}} \to \epsilon^{-1} \Omega_{\mathcal{M}}, \quad \gamma \to \epsilon \gamma$$

With $L := b_R b_R^\dagger$ and $b_R = e^{\epsilon p} b_+(Q, \epsilon^{\frac{1}{2}} W)$, we find

$$\mathrm{tr}(L^{\pm 1}) = n \pm 2\epsilon\, \mathrm{tr}(p) + 2\epsilon^2 \mathrm{tr}(p^2) + \epsilon^2 \sum_{i<j} \frac{|(w_i^\bullet, w_j^\bullet)|^2}{|Q_j Q_i^{-1} - 1|^2} + \mathrm{o}(\epsilon^2)$$

where $w_i^\bullet \in \mathbb{C}^d$ with components w_i^α, and $(w_i^\bullet, w_j^\bullet) := \sum_{\alpha=1}^d w_i^\alpha \overline{w}_j^\alpha$. Writing $Q_j = e^{\mathrm{i} q_j}$, we obtain on $\mathcal{M}_0^{\mathrm{reg}}$

$$\lim_{\epsilon \to 0} \frac{1}{8\epsilon^2} (\mathrm{tr}(L) + \mathrm{tr}(L^{-1}) - 2n) = \frac{1}{2} \mathrm{tr}(p^2) + \frac{1}{32} \sum_{i \neq j} \frac{|(w_i^\bullet, w_j^\bullet)|^2}{\sin^2 \frac{q_i - q_j}{2}},$$

which reproduces Hamiltonian of the (real, trigonometric) Gibbons–Hermsen model, and

$$\lim_{\epsilon \to 0} \epsilon^{-1} (\Omega_{\mathcal{M}}) = \sum_{j=1}^n dp_j \wedge dq_j + \frac{\mathrm{i}}{2} \sum_{j=1}^n \sum_{\alpha=1}^d dw_j^\alpha \wedge d\overline{w}_j^\alpha,$$

which reproduces the symplectic form of the Gibbons–Hermsen model.

Now we describe the commuting vector fields generated by the Hamiltonians $H_m := \frac{1}{2m}\mathrm{tr}(L^m)$ for $m = 1, \dots, n$. Before reduction, we parametrize the phase space by the variables (g_R, L, v), where $L = b_R b_R^\dagger$ and $v = (v(1), v(2), \dots, v(d)) \in \mathbb{C}^{n\times d}$ denotes the 'dressed spins'.

The Hamiltonian vector field X_{H_m} reads

$$X_{H_m}[g_R] = \mathrm{i}L^m g_R, \quad X_{H_m}[v(\alpha)] = 0, \quad X_{H_m}[L] = 0.$$

Its projection onto $\mathcal{M}_{\mathrm{red}}$ does not change if we add any infinitesimal gauge transformation, i.e., consider Y_{H_m} given by

$$\begin{aligned} Y_{H_m}[g_R] &= \mathrm{i}L^m g_R + [Z(g_R, L, v), g_R], \\ Y_{H_m}[v(\alpha)] &= Z(g_R, L, v) v(\alpha), \\ Y_{H_m}[L] &= [Z(g_R, L, v), L], \end{aligned}$$

with arbitrary $Z(g_R, L, v) \in \mathfrak{u}(n)$. To determine the projection, one may use the restriction of Y_{H_m} to

$$\mathcal{M}_0 \equiv \Lambda^{-1}(e^\gamma \mathbf{1}_n) \cap \Xi_R^{-1}(\mathbb{T}^n) \quad \text{where} \quad g_R := Q := e^{\mathrm{i}q} \in \mathbb{T}^n.$$

Proposition 1. If $(Q, L, v) \in \mathcal{M}_0$, then L can be expressed in terms of Q and v as follows:

$$L_{ij} = \frac{F_{ij}}{e^{2\gamma} Q_j Q_i^{-1} - 1} \quad \text{with} \quad F := \sum_{\alpha=1}^{d} v(\alpha) v(\alpha)^\dagger.$$

Conversely, if the Hermitian matrix L given by the above formula is positive definite, then $(Q, L, v) \in \mathcal{M}_0$. Thus, $\mathcal{M}_0$ is identified with an open subset of $\mathbb{T}^n \times \mathbb{C}^{n \times d}$.

We focus on the regular part, and choose $Z(Q, L, v) := \mathcal{K}_m(Q, L)$ with $\mathcal{K}_m(Q, L)_{kk} = 0$ and

$$\mathcal{K}_m(Q, L)_{kl} = -\frac{1}{2}\mathrm{i}(L^m)_{kl} - \frac{\mathrm{i}}{2}\mathrm{i}(L^m)_{kl} \cot\left(\frac{q_k - q_l}{2}\right), \quad \forall k \neq l.$$

which guarantees tangency of the restricted vector field, $Y^0_{H_m}$, to $\mathcal{M}_0^{\mathrm{reg}}$.

One may add an arbitrary $\mathrm{Lie}(\mathbb{T}^n)$-valued function $\lambda(Q, v)$ to $\mathcal{K}_m$, expressing the residual infinitesimal gauge transformations.

Proposition 2. $H_m := \frac{1}{2m}\mathrm{tr}(L^m)$ induces the vector field $Y^0_{H_m}$ on $\mathcal{M}_0^{\mathrm{reg}}$,

$$Y^0_{H_m}[Q] = \mathrm{i}(L^m)_{\mathrm{diag}}Q$$
$$Y^0_{H_m}[v(\alpha)] = \mathcal{K}_m(Q,L)v(\alpha)$$
$$Y^0_{H_m}[L] = [\mathcal{K}_m(Q,L),L],$$

which descends to the Hamiltonian vector field of the corresponding reduced Hamiltonian on $\mathcal{M}_{\mathrm{red}}^{\mathrm{reg}} \subset \mathcal{M}_{\mathrm{red}}$.

Corollary. Consider $H := (e^{2\gamma}-1)\mathrm{tr}(L)$. Then the evolution equation on $\mathcal{M}_0^{\mathrm{reg}}$ corresponding to the vector field Y^0_H can be written as follows:

$$\frac{1}{2}\dot{q}_j := \frac{1}{2\mathrm{i}}Y^0_H[Q_j]Q_j^{-1} = F_{jj},$$
$$\dot{v}(\alpha)_i := Y^0_H[v(\alpha)_i] = -\sum_{j\neq i} F_{ij}v(\alpha)_j V\left(\frac{q_j-q_i}{2}\right)$$

with the 'potential function' $V(x) = \cot x - \cot(x-\mathrm{i}\gamma)$. This reproduces the spin RS equations of motion of Krichever and Zabrodin by setting $x_i = q_i/2$ and imposing the reality conditions $c_i^\alpha = (a_i^\alpha)^* \equiv v(\alpha)_i$.

We have also calculated the reduced Poisson bracket in terms of the variables $Q = e^{\mathrm{i}q}, v$ that appear in the reduced equations on motion. More precisely, we did this on an S_n covering space of a dense open subset of $\mathcal{M}_{\mathrm{red}}$. Our model of this dense open subset is given by

$$\breve{\mathcal{M}}_{0,+}^{\mathrm{reg}} := \{(Q, L(Q,v), v) \in \mathcal{M}_0^{\mathrm{reg}} \mid \sum_{1\leq\alpha\leq d} v(\alpha)_i > 0 \text{ for } i = 1, \ldots, n\}.$$

That is, we assumed that the all components of the $\mathbb{C}^n$-vector

$$\mathcal{U}_i := \sum_{1\leq\alpha\leq d} v(\alpha)_i$$

are non-zero, and fixed the $\mathbb{T}^n$ gauge freedom by constraining them to be real and positive.

We proceeded by calculating the Poisson brackets of the following $\mathrm{U}(n)$ invariant functions on the unreduced phase space $\mathcal{M}$:

$$f_m^{\alpha\beta} := \mathrm{tr}(v(\alpha)v(\beta)^\dagger g_R^m) = v(\beta)^\dagger g_R^m v(\alpha)\,, \qquad f_m := \mathrm{tr}(g_R^m).$$

They reduce to $f_m^{\alpha\beta} = \sum_{i=1}^n v(\alpha)_i Q_i^m \overline{v}(\beta)_i$ and $f_m = \sum_{i=1}^n Q_i^m$, and their reduced Poisson brackets determine those of q_i and $v(\alpha)_j$ on $\breve{\mathcal{M}}_{0,+}^{\mathrm{reg}}$.

We obtain the simple reduced Poisson brackets

$$\{q_i, q_j\}_{\text{red}} = 0\,, \quad \{v(\alpha)_i, q_j\}_{\text{red}} = -\delta_{ij} v(\alpha)_i\,,$$

but the formulae for $\{v(\alpha)_i, v(\beta)_j\}_{\text{red}}$ and $\{v(\alpha)_i, \overline{v}(\beta)_j\}_{\text{red}}$ are rather involved. These reduced Poisson brackets enjoy a residual S_n symmetry, and the PBs of the S_n invariant functions descend to $\check{\mathcal{M}}^{\text{reg}}_{\text{red}} \equiv \check{\mathcal{M}}^{\text{reg}}_{0,+}/S_n$.

We have checked that reduced Hamiltonian

$$\mathcal{H}(Q, v) = (e^{2\gamma} - 1)\,\text{tr}(L(Q, v)) = \sum_{k=1}^{n} \sum_{\alpha=1}^{d} |v(\alpha)_k|^2$$

generates the correct 'projected Hamiltonian vector field' on the gauge slice $\check{\mathcal{M}}^{\text{reg}}_{0,+}$, as it must. We also determined the Poisson brackets of the Lax matrix $L(Q, v)$, which turned out to have the form

$$\{L_1, L_2\}_{\text{red}} = r_{12} L_1 L_2 + L_1 L_2 t_{12} - L_1 s_{21} L_2 + L_2 s_{12} L_1\,,$$

where $t_{12} = -s_{12} + s_{21} - r_{12}$, and both r and s are 'fully dynamical'. This is consistent with the fact that the functions $\text{tr}(L^k)$ are in involution. (For explicit formulae, see Theorem 5.8 and Proposition 5.10. in our paper.)

The reduced dynamics is 'solvable by algebraic manipulations'. We finish by sketching its degenerate integrability, i.e., the construction of sufficient number of integrals of motion. For this, we consider the polynomial subalgebra of $C^\infty(\mathcal{M})^{\mathrm{U}(n)}$:

$$\mathcal{I}_L = \mathbb{R}[\mathrm{tr}L^k, \Re(I^k_{\alpha\beta}), \Im(I^k_{\alpha\beta}) \mid 1 \leq \alpha, \beta \leq d,\, k \geq 0], \quad I^k_{\alpha\beta} := \mathrm{tr}\left(v(\alpha)v(\beta)^\dagger L^k\right).$$

This is closed under the Poisson bracket and its center contains

$$\mathfrak{H}_{\mathrm{tr}} := \mathbb{R}[\mathrm{tr}L^k, k \geq 0]\,.$$

Explicitly, we have

$$\begin{aligned}\{I^M_{\alpha\beta}, I^N_{\gamma\epsilon}\} = {} & 2\mathrm{i}\delta_{\alpha\epsilon} I^{M+N+1}_{\gamma\beta} - 2\mathrm{i}\delta_{\gamma\beta} I^{M+N+1}_{\alpha\epsilon} \\ & + \mathrm{i}(\delta_{\alpha\epsilon} - \delta_{\gamma\beta}) I^M_{\alpha\beta} I^N_{\gamma\epsilon} + 2\mathrm{i}\delta_{\alpha\epsilon} \sum_{\mu<\alpha} I^N_{\gamma\mu} I^M_{\mu\beta} - 2\mathrm{i}\delta_{\gamma\beta} \sum_{\lambda<\beta} I^M_{\alpha\lambda} I^N_{\lambda\epsilon} \\ & + \mathrm{i}\,\mathrm{sgn}(\gamma - \alpha) I^M_{\gamma\beta} I^N_{\alpha\epsilon} - \mathrm{i}\,\mathrm{sgn}(\epsilon - \beta) I^N_{\gamma\beta} I^M_{\alpha\epsilon} \\ & + \mathrm{i}\left(\sum_{b=0}^{M-1} + \sum_{b=0}^{N-1}\right)\left(I^b_{\gamma\beta} I^{M+N-b}_{\alpha\epsilon} - I^{M+N-b}_{\gamma\beta} I^b_{\alpha\epsilon}\right)\end{aligned}$$

and the reality property $\{\overline{I^M_{\alpha\beta}}, \overline{I^N_{\gamma\epsilon}}\} = \overline{\{I^M_{\alpha\beta}, I^N_{\gamma\epsilon}\}}$.

Our Hamiltonian reduction actually works in the real-analytic category, and $\mathfrak{H}_{\mathrm{tr}}$ and $\mathcal{I}_L$ descend to polynomial Poisson algebras on the connected, real-analytic reduced symplectic manifold $(\mathcal{M}_{\mathrm{red}}, \Omega_{\mathrm{red}})$.

Theorem 2. The reduced polynomial algebras of functions $\mathfrak{H}_{\mathrm{tr}}^{\mathrm{red}}$ and $\mathcal{I}_L^{\mathrm{red}}$ inherited from $\mathfrak{H}_{\mathrm{tr}}$ and $\mathcal{I}_L$ have functional dimension n and $2nd - n$, respectively. In particular, on the phase space $\mathcal{M}_{\mathrm{red}}$ of dimension $2nd$, the Abelian Poisson algebra $\mathfrak{H}_{\mathrm{tr}}^{\mathrm{red}}$ yields a real-analytic, degenerate integrable system with integrals of motion $\mathcal{I}_L^{\mathrm{red}}$

Concretely, for any $d > 1$, we proved that the $2n(d-1)$ integrals of motion:

$$\mathrm{tr}(L^k), \quad I_{1,1}^k, \quad \Re[I_{\alpha,1}^k], \quad \Im[I_{\alpha,1}^k]$$

with $k = 1, \ldots, n$ and $\alpha = 2, \ldots, d-1$, are independent after reduction, and n further integrals of motion may be selected from the real and imaginary parts of the functions $I_{d,1}^k$ in such a way that all in all these provide a set of $2nd - n$ independent functions.

In the $d = 1$ case $\mathfrak{H}_{\mathrm{tr}}^{\mathrm{red}} = \mathfrak{H}_L^{\mathrm{red}}$ and one has (only) Liouville integrability.

To conclude, we have shown that the trigonometric real form of the spin RS system of Krichever and Zabrodin arises from Hamiltonian reduction of a 'free system' on a spin extension of the Heisenberg double of U(n).

We gave two models of open dense subsets of the reduced phase space. One is convenient for characterizing the reduced symplectic form and for making contact with the Gibbons–Hermsen model. The other one reproduces the K-Z equations of motion and gives their Hamiltonian structure. We have proved degenerate integrability by displaying the required constants of motion, and their Poisson algebra.

However, a global model of the reduced phase space was not obtained.

Some further open problems: Can one derive a dual system, for which the commuting Hamiltonians should arise from $\Xi_R^*\Big(C^\infty(\mathrm{U}(n))^{\mathrm{U}(n)}\Big)$? Can one construct compactified versions? What about the hyperbolic real form? Quantization? Elliptic systems?

For details and refs, see our paper, arXiv:2007.08388 by M. Fairon, L.F. and I. Marshall, which just appeared in Annales Henri Poincaré.