Applications of Hamiltonian reduction to integrable many-body systems

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• Tenet: Integrable systems are "shadows" of "free systems".

• Two integrable many-body models are dual to each other if the action variables of model-1 are the particle coordinates of model-2, and vice versa. Self-duality occurs as a special case.

• Duality was originally discovered by Ruijsenaars (1988-95) in his direct construction of action-angle variables for Calogero-Sutherland type models and their relativistic generalizations.

• First illustration: derive the hyperbolic Sutherland and the rational Ruijsenaars-Schneider models by a *single* reduction of certain 'canonical free systems', which will explain their duality.

• Second illustration: derive action-angle map for open Toda lattice.

The simplest self-dual system: $\quad H_{\mathrm{Cal}}(q,p) = \frac{1}{2}\sum_{k=1}^{n} p_k^2 + \frac{1}{2}\sum_{j\neq k} \frac{x^2}{(q_k - q_j)^2}$

Symplectic reduction: Consider phase space $T^*\mathrm{i}u(n) \simeq \mathrm{i}u(n) \times \mathrm{i}u(n) := \{(Q,P)\}$ with two families of 'free' Hamiltonians $\{\mathrm{tr}\,(Q^k)\}$ and $\{\mathrm{tr}\,(P^k)\}$. Reduce by the adjoint action of $U(n)$ using the moment map constraint

$$[Q,P] = \mu(x) := \mathrm{i}x \sum_{j\neq k} E_{j,k}$$

This yields the self-dual Calogero system (OP [76], KKS [78]):

gauge slice (i): $Q = q := \mathrm{diag}(q_1,\dots,q_n)$, $\; q_1 > \cdots > q_n$, with $p := \mathrm{diag}(p_1,\dots,p_n)$

$$P = p + \mathrm{i}x \sum_{j\neq k} \frac{E_{jk}}{q_j - q_k} \equiv L_{\mathrm{Cal}}(q,p) \quad \text{Lax matrix}, \quad \mathrm{tr}\,(dP \wedge dQ) = \sum_{k=1}^{n} dp_k \wedge dq_k$$

gauge slice (ii): $P = \hat{p} := \mathrm{diag}(\hat{p}_1,\dots,\hat{p}_n)$, $\; \hat{p}_1 > \cdots > \hat{p}_n$, with $\hat{q} := \mathrm{diag}(\hat{q}_1,\dots,\hat{q}_n)$

$$Q = -L_{\mathrm{Cal}}(\hat{p},\hat{q}) \quad \text{dual Lax matrix}, \quad \mathrm{tr}\,(dP \wedge dQ) = \sum_{k=1}^{n} d\hat{q}_k \wedge d\hat{p}_k.$$

The alternative gauge slices give two models of the reduced phase space. Their natural symplectomorphism is the self-duality map.

For a recent application, see T.F. Gorbe: A simple proof of Sklyanin's formula for canonical spectral coordinates of the rational Calogero-Moser system, arXiv:1601.01181

First dual pair of many-body models

The hyperbolic Sutherland model (1971):

$$H_{\mathsf{hyp-Suth}}(q,p) \equiv \frac{1}{2}\sum_k p_k^2 + \frac{\kappa^2}{2}\sum_{j\neq k}\frac{1}{\sinh^2(q^j - q^k)}$$

Basic Poisson brackets: $\{q^i, p_j\} = \delta^i_j$.

The rational Ruijsenaars-Schneider model (1986):

$$H_{\mathsf{rat-RS}}(\hat{p},\hat{q}) \equiv \sum_k \cosh(\hat{q}_k)\prod_{j\neq k}\left[1 + \frac{\kappa^2}{(\hat{p}^k - \hat{p}^j)^2}\right]^{\frac{1}{2}}$$

Basic Poisson brackets: $\{\hat{p}^i, \hat{q}_j\} = \delta^i_j$ ($\hat{p}^i$ are RS ‘coordinates’)

Models describe n ‘particles’ moving on the line, and are integrable (exhibit factorizable scattering).

Canonical integrable systems

Consider real Lie algebra $\mathcal{G} := gl(n, \mathbb{C})$ with bilinear form

$$\langle X, Y \rangle := \Re\text{tr}\,(XY) \qquad \forall X, Y \in \mathcal{G},$$

and Lie group $G := GL(n, \mathbb{C})$. **Phase space** is cotangent bundle

$$T^*G \simeq G \times \mathcal{G} = \{(g, J^R) \,|\, g \in G,\ J^R \in \mathcal{G}\}$$

with **symplectic form**

$$\Omega = d\langle J^R, g^{-1}dg \rangle$$

In terms of local coordinates x^a and momenta π_a: $\Omega = \sum_a d\pi_a \wedge dx^a$

With basis $\{T_a\}$ of $\mathcal{G}$, the basic **Poisson brackets** are

$$\{g_{jk}, \langle J^R, T_a \rangle\} = (gT_a)_{jk}, \quad \{\langle J^R, T_a \rangle, \langle J^R, T_b \rangle\} = -\langle J^R, [T_a, T_b] \rangle$$

and any two functions of 'configuration space' variable g commute.

Introduce matrix functions $\mathcal{L}_1$ and $\mathcal{L}_2$ on T^*G by

$$\mathcal{L}_1(g, J^R) := J^R \quad \text{and} \quad \mathcal{L}_2(g, J^R) := gg^\dagger$$

These 'unreduced Lax matrices' generate 'canonical Hamiltonians'

$$H_j := \frac{1}{j}\Re\mathrm{tr}\,(\mathcal{L}_1^j), \qquad j = 1, \ldots, n$$

$$\hat{H}_k := \frac{1}{2k}\mathrm{tr}\,(\mathcal{L}_2^k), \qquad k = \pm 1, \ldots, \pm n$$

- Both $\{H_j\}$ and $\{\hat{H}_k\}$ form Abelian algebras.

- One can write down their Hamiltonian flows explicitly.

- They are invariant under large symmetry group.

Interesting models are reductions of 'obviously integrable' systems.

Hamiltonian flow defined by H_j:

$$g(t) = g(0)\exp(t(J^R(0))^{j-1}), \qquad J^R(t) = J^R(0).$$

Flow generated by $\hat{H}_k$:

$$J^R(t) = J^R(0) - t\left(g^\dagger(0)g(0)\right)^k, \quad g(t) = g(0).$$

We shall reduce by symmetry group

$$K := U(n)^L \times U(n)^R$$

$(\eta_L, \eta_R) \in K$ ($\eta_{L,R} \in U(n)$) acts by 'canonical transformation' Ψ_{η_L,η_R},

$$\Psi_{\eta_L,\eta_R} : (g, J^R) \mapsto (\eta_L g \eta_R^{-1}, \eta_R J^R \eta_R^{-1})$$

'Infinitesimal generators' of symmetry are given by 'moment map'

$$\Phi : T^*G \to u(n)^L \oplus u(n)^R, \qquad \Phi(g, J^R) = ((gJ^R g^{-1})_+, -J^R_+)$$

Here, $\forall X \in \mathcal{G} : X = X_+ + X_-$ with $X_+ \in u(n),\ X_- \in \mathrm{i}u(n)$

- Hamiltonians H_j and $\hat{H}_k$ are invariant under symmetry group K.
- Φ is constant of motion for flows of H_j and $\hat{H}_k$.

Steps of the reduction procedure:

1. Fix the conserved quantities encoded by Φ to some constant (in other words: introduce constraints on phase space).

2. Factorize (that is: eliminate variables) by ‘residual symmetry transformations’: symmetries preserving the chosen value of Φ.

Result: Reduced phase space with Abelian algebras induced by $\{H_j\}$ and $\{\widehat{H}_k\}$.

The reduced systems can be solved by ‘projecting’ the original flows.

The *art* is to find ‘good value’ of the constants of motion.

Paradigm: Fix angular momentum in spherically symmetric problem and factorize out angle corresponding to rotations around the fixed angular momentum. The reduced system will be a ‘radial equation’.
Here: want to solve ‘radial equation’ by viewing it as reduction of ‘trivial problem’.

Our choice of moment map constraint:

$$J_+^R = 0, \qquad (gJ^R g^{-1})_+ = \mu_\kappa := \mathrm{i}\kappa(\mathbf{1}_n - ww^\dagger)$$

with real constant κ and vector $w^\dagger := (1, 1, \ldots, 1)$.

For technical convenience, we introduce 'extended phase space' where extended moment map will be set to zero, giving same result.

$$\text{Define} \quad \mathcal{O}_\kappa^L := \{\xi = \mathrm{i}\kappa(vv^\dagger - \mathbf{1}_n) \,|\, v \in \mathbb{C}^n,\ |v|^2 = n\}.$$

Elements $\xi \in \mathcal{O}_\kappa^L$ are of the form $-\eta\mu_\kappa\eta^{-1}$ with $\eta \in U(n)$. $\mathcal{O}_\kappa^L$ is orbit of $U(n)$ with natural symplectic form, $\Omega^{\mathcal{O}}$, and Poisson bracket

$$\{\langle\xi, T\rangle, \langle\xi, V\rangle\} = \langle\xi, [T, V]\rangle \qquad \forall T, V \in u(n).$$

On extended phase space $T^*G \times \mathcal{O}_\kappa^L = \{(g, J^R, \xi)\}$, symmetry group K acts by $\Psi^{\mathrm{ext}}_{\eta_L,\eta_R} : (g, J^R, \xi) \mapsto (\eta_L g \eta_R^{-1}, \eta_R J^R \eta_R^{-1}, \eta_L \xi \eta_L^{-1})$.

Infinitesimal generator is $\Phi^{\mathrm{ext}}(g, J^R, \xi) = ((gJ^R g^{-1})_+ + \xi, -J_+^R)$.

We reduce by imposing $\Phi^{\mathrm{ext}} = 0$, and then factorizing by K.

Extended canonical integrable systems

Before reduction, we extend 'canonical Hamiltonians' to $T^*G \times \mathcal{O}^L_\kappa$ by declaring that they do not depend on 'auxiliary variable' $\xi \in \mathcal{O}^L_\kappa$:

$$H_j^{\text{ext}}(g, J^R, \xi) := H_j(g, J^R), \qquad H_k^{\text{ext}}(g, J^R, \xi) := \hat{H}_k(g, J^R)$$

Flows on $T^*G \times \mathcal{O}^L_\kappa$ are same as flows on T^*G adding $\xi(t) = \xi(0)$.

Extended Hamiltonians are spectral invariants of

$$\mathcal{L}_1^{\text{ext}}(g, J^R, \xi) = J^R \quad \text{and} \quad \mathcal{L}_2^{\text{ext}}(g, J^R, \xi) = gg^\dagger,$$

since $H_j^{\text{ext}} = \frac{1}{j}\Re\text{tr}\left((\mathcal{L}_1^{\text{ext}})^j\right)$ and $\hat{H}_k^{\text{ext}} = \frac{1}{2k}\text{tr}\left((\mathcal{L}_2^{\text{ext}})^k\right)$.

In general, Lax matrices matter only up to similarity transformation. Now

$$\Psi^{\text{ext}}_{\eta_L,\eta_R} : \mathcal{L}_1^{\text{ext}} \mapsto \eta_R \mathcal{L}_1^{\text{ext}} \eta_R^{-1}, \qquad \Psi^{\text{ext}}_{\eta_L,\eta_R} : \mathcal{L}_2^{\text{ext}} \mapsto \eta_L \mathcal{L}_2^{\text{ext}} \eta_L^{-1}.$$

Therefore, the reduced Hamiltonians will be generated by reduced Lax matrices.

Definition of the reduced systems

Reduced phase space is space of K-orbits in level set $\Phi^{\text{ext}} = 0$:

$$T^*G \times \mathcal{O}_\kappa^L //_0 K \equiv (\Phi^{\text{ext}})^{-1}(0)/K$$

In our case this is a smooth manifold, as we shall see.

Using the natural injection and projection maps

$$\iota : (\Phi^{\text{ext}})^{-1}(0) \to T^*G \times \mathcal{O}_\kappa^L, \quad \pi : (\Phi^{\text{ext}})^{-1}(0) \to (\Phi^{\text{ext}})^{-1}(0)/K$$

reduced symplectic form, Ω^{red}, is characterized by

$$\pi^*\Omega^{\text{red}} = \iota^*\Omega^{\text{ext}} \quad \text{with} \quad \Omega^{\text{ext}} = \Omega + \Omega^{\mathcal{O}}$$

In another language, Ω^{red} encodes the so-called Dirac bracket.

Reduced Hamiltonians H_j^{red} and $\hat{H}_k^{\text{red}}$ are defined by

$$H_j^{\text{red}} \circ \pi = H_j^{\text{ext}} \circ \iota, \qquad H_k^{\text{red}} \circ \pi = \hat{H}_k^{\text{ext}} \circ \iota$$

Next, we shall present **two** models of **the** reduced phase space.

Notationwise, associate to any vector $q \in \mathbb{R}^n$ the diagonal matrix

$$\mathbf{q} := \text{diag}(q^1, \dots, q^n).$$

Let $\mathcal{C}$ denote the open domain (Weyl chamber)

$$\mathcal{C} := \{ q \in \mathbb{R}^n \,|\, q^1 > q^2 > \cdots > q^n \}.$$

Equip $T^*\mathcal{C} \simeq \mathcal{C} \times \mathbb{R}^n = \{(q,p)\}$ with the Darboux form

$$\Omega_{T^*\mathcal{C}}(q,p) := \sum_k dp_k \wedge dq^k$$

corresponding to the canonical Poisson bracket.

Define $\mathrm{i}u(n)$-valued (Hermitian) matrix function L_1 on $T^*\mathcal{C}$ by

$$L_1(q,p)_{jk} := p_j \delta_{jk} - \mathrm{i}(1-\delta_{jk}) \frac{\kappa}{\sinh(q^j - q^k)}$$

L_1 is actually the standard Lax matrix of the Sutherland model.

First model: the Sutherland gauge slice S_1

Theorem 1. *The manifold S_1 defined by*

$$S_1 := \{ (e^{\mathbf{q}}, L_1(q,p), -\mu_\kappa) \,|\, (q,p) \in \mathcal{C} \times \mathbb{R}^n \}$$

is a **global cross section** *of the K-orbits in the submanifold* $(\Phi^{\text{ext}})^{-1}(0)$ *of $T^*G \times \mathcal{O}^L_\kappa$. If $\iota_1 : S_1 \to T^*G \times \mathcal{O}^L_\kappa$ is the obvious injection, then in terms of the coordinates q, p on S_1 one has*

$$\iota_1^*(\Omega^{\text{ext}}) = \sum_k dp_k \wedge dq^k.$$

That is, the Dirac bracket on S_1 is just the canonical Poisson bracket $\{q^i, p_j\} = \delta^i_j$.

Therefore, the symplectic manifold

$$(S_1, \sum_k dp_k \wedge dq^k) \simeq (T^*\mathcal{C}, \Omega_{T^*\mathcal{C}})$$

is a model of the reduced phase space.

Theorem 1 due to Olshanetsky-Perelomov [76], Kazhdan-Kostant-Sternberg [78].

Next, denote the elements of $T^*\mathcal{C} = \mathcal{C} \times \mathbb{R}^n$ as pairs $(\widehat{p}, \widehat{q})$.

Define $n \times n$ (Hermitian, positive definite) matrix-valued function L_2 on $T^*\mathcal{C}$ by

$$L_2(\widehat{p}, \widehat{q})_{jk} = u_j(\widehat{p}, \widehat{q}) \left[\frac{\mathrm{i}\kappa}{\mathrm{i}\kappa + (\widehat{p}^j - \widehat{p}^k)} \right] u_k(\widehat{p}, \widehat{q})$$

with

$$u_j(\widehat{p}, \widehat{q}) := e^{-\widehat{q}_j/2} \prod_{m \neq j} \left[1 + \frac{\kappa^2}{(\widehat{p}^j - \widehat{p}^m)^2} \right]^{\frac{1}{4}}, \quad j = 1, \dots, n.$$

Then define $\mathbb{R}^n$-valued function

$$v(\widehat{p}, \widehat{q}) := L_2(\widehat{p}, \widehat{q})^{-\frac{1}{2}} u(\widehat{p}, \widehat{q}),$$

where $u = (u_1, \dots, u_n)^T$. It can be verified that $|v(\widehat{p}, \widehat{q})|^2 = n$.

Finally, introduce the $\mathcal{O}_\kappa^L$-valued function

$$\xi(\widehat{p}, \widehat{q}) := \xi(v(\widehat{p}, \widehat{q})) = \mathrm{i}\kappa(v(\widehat{p}, \widehat{q}) v(\widehat{p}, \widehat{q})^\dagger - \mathbf{1}_n)$$

L_2 is actually the standard Lax matrix of the Ruijsenaars-Schneider model.

Second model: the Ruijsenaars gauge slice S_2

Theorem 2. *The manifold S_2 defined by*

$$S_2 := \{ \, (L_2(\hat{p},\hat{q})^{\frac{1}{2}}, 2\hat{\mathbf{p}}, \xi(\hat{p},\hat{q})) \, | \, (\hat{p},\hat{q}) \in \mathcal{C} \times \mathbb{R}^n \, \}$$

is a **global cross section** *of the K-orbits in the submanifold* $(\Phi^{\rm ext})^{-1}(0)$ *of $T^*G \times \mathcal{O}^L_\kappa$. If $\iota_2 : S_2 \to T^*G \times \mathcal{O}^L_\kappa$ is the obvious injection, then in terms of the coordinates $\hat{p}$, $\hat{q}$ on S_2 one has*

$$\iota_2^*(\Omega^{\rm ext}) = \sum_k d\hat{q}_k \wedge d\hat{p}^k.$$

That is, the Dirac bracket on S_2 is just the canonical Poisson bracket $\{\hat{p}^i, \hat{q}_j\} = \delta^i_j$.

Therefore, the symplectic manifold

$$(S_2, \sum_k d\hat{q}_k \wedge d\hat{p}^k) \simeq (T^*\mathcal{C}, \Omega_{T^*\mathcal{C}})$$

is a model of the reduced phase space.

Theorem 2 is the main result of L.F.-C. Klimčík: J. Phys. A: Math. Theor. 42 (2009) 185202

Consequences

1. Since S_1 and S_2 are two models of **the** reduced phase space, there exists a natural canonical transformation (symplectomorphism) between these two models:

$$(S_1, \sum_k dp_k \wedge dq^k) \equiv (T^*G \times \mathcal{O}^L_\kappa //_0 K, \Omega^{\text{red}}) \equiv (S_2, \sum_k d\widehat{q}_k \wedge d\widehat{p}^k).$$

By definition, a point of S_1 is related to that point of S_2 which represents the same element of the reduced phase space.

2. The K-invariant Hamiltonians H_j^{ext} and $\widehat{H}_k^{\text{ext}}$ descend to the reduced Hamiltonians $\{H_j^{\text{red}}\}$ and $\{\widehat{H}_k^{\text{red}}\}$ on $T^*G \times \mathcal{O}_\kappa^L//_0 K$, whose commutativity follows from the construction. The restrictions of the 'unreduced Lax matrices' to S_1 and S_2 satisfy

$$\mathcal{L}_1^{\text{ext}}|_{S_1} = L_1 \quad \text{and} \quad \mathcal{L}_2^{\text{ext}}|_{S_2} = L_2.$$

The reduced Hamiltonians take following form in terms of the 'gauge slices' $(S_1, \sum_k dp_k \wedge dq^k)$ and $(S_2, \sum_k d\widehat{q}_k \wedge d\widehat{p}^k)$:

$$\text{on } S_1: \qquad H_j^{\text{red}} = \frac{1}{j}\text{tr}\,(L_1^j)\,, \qquad \widehat{H}_k^{\text{red}} = \frac{1}{2k}\sum_{i=1}^n (e^{2q^i})^k$$

$$\text{on } S_2: \qquad H_j^{\text{red}} = \frac{1}{j}\sum_{i=1}^n (2\widehat{p}^i)^j\,, \qquad \widehat{H}_k^{\text{red}} = \frac{1}{2k}\text{tr}\,(L_2^k)$$

3. L_1 is the Lax matrix of the hyperbolic Sutherland model and L_2 is the Lax matrix of the rational Ruijsenaars-Schneider model. Indeed, the basic Hamiltonians of these models are

$$H_{\text{hyp}-\text{Suth}}(q,p) \equiv \frac{1}{2}\sum_k p_k^2 + \frac{\kappa^2}{2}\sum_{j\neq k}\frac{1}{\sinh^2(q^j - q^k)} = \frac{1}{2}\text{tr}\left(L_1(q,p)^2\right)$$

$$H_{\text{rat}-\text{RS}}(\hat{p},\hat{q}) \equiv \sum_k \cosh(\hat{q}_k)\prod_{j\neq k}\left[1+\frac{\kappa^2}{(\hat{p}^k - \hat{p}^j)^2}\right]^{\frac{1}{2}} = \frac{1}{2}\text{tr}\left(L_2(\hat{p},\hat{q}) + L_2(\hat{p},\hat{q})^{-1}\right)$$

Besides the Hamiltonians, also the Lax matrices arose naturally from the reduction.

4. Consider two points of S_1 and S_2 that lie on the same K-orbit, and are parametrized by some $(q,p) \in \mathcal{C} \times \mathbb{R}^n$ and by $(\hat{p},\hat{q}) \in \mathcal{C} \times \mathbb{R}^n$.

Then there exists $\eta \in U(n)$ for which

$$(\eta e^{\mathbf{q}} \eta^{-1}, \eta L_1(q,p)\eta^{-1}, -\eta \mu_\kappa \eta^{-1}) = (L_2(\hat{p},\hat{q})^{\frac{1}{2}}, 2\hat{\mathbf{p}}, \xi(\hat{p},\hat{q})).$$

Therefore:

The matrix $2\hat{\mathbf{p}}$, which encodes coordinate-variables of rational RS model, results by diagonalizing the Sutherland Lax matrix $L_1(q,p)$.

Conversely, $e^{2\mathbf{q}}$, which encodes coordinate-variables of Sutherland model, results by diagonalizing the RS Lax matrix $L_2(\hat{p},\hat{q})$.

This reproduces, effortlessly, original direct construction due to Ruijsenaars (1988).

5. Now it is *obvious* that the two many-body models are dual to each other.

On the one hand, the Ruijsenaars-Schneider particle-coordinates $\widehat{p}^1, \ldots, \widehat{p}^n$ *regarded as functions on* S_1 define action variables for the hyperbolic Sutherland model.

On the other, the Sutherland particle coordinates $q^1, \ldots, q^n$ *regarded as functions on* S_2 can serve as action variables for the rational Ruijsenaars-Schneider model.

6. The known solution algorithms for the commuting Hamiltonians of the models are easy byproducts of the geometric approach.

First, take an initial value on the 'gauge slice' S_1 and project the 'free flow' of H_j^{ext} back to S_1. This implies that H_j^{red} generates the following evolution for the Sutherland coordinate variables:

$$e^{2\mathbf{q}(t)} = \mathcal{D}[e^{\mathbf{q}(0)} \exp(2tL_1(0)^{j-1})e^{\mathbf{q}(0)}],$$

where $\mathcal{D}$ is the operator that brings its Hermitian matrix-argument to diagonal form with eigenvalues in non-increasing order.

Similarly, we obtain that $\hat{H}_k^{\text{red}}$ generates the following flow for the Ruijsenaars-Schneider coordinate variables:

$$2\hat{\mathbf{p}}(t) = \mathcal{D}[2\hat{\mathbf{p}}(0) - tL_2(0)^k].$$

The particles move as eigenvalues of 'geodesic in a space of matrices', as usual.

Conclusions so far and plan of what follows

We interpreted the duality between the hyperbolic Sutherland and the rational Ruijsenaars-Schneider models in geometric terms.

Thus we obtained a Lie theoretic understanding of results due to Ruijsenaars (88), who discovered the duality 'by bare hands'.

Our symplectic reduction approach simplifies a considerable portion of the original technical arguments. It can be – and was – adapted to explore more complicated cases of the duality, too.

Next, I present a group-theoretic interpretation of old results about action-angle map and duality for open Toda

$$H_{\mathrm{Toda}}(q,p) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

(based on Phys. Lett. A 377, 2917-2921 (2013))

Ruijsenaars (1990) found explicit action-angle map for Toda Hamiltonian system (M, ω, H) and introduced dual integrable system.

$$M := \mathbb{R}^n \times \mathbb{R}^n = \{(q,p)\}, \quad \omega = \sum_{i=1}^n dp_i \wedge dq_i, \quad H = \frac{1}{2}\sum_{i=1}^n p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}$$

Phase space of action-angle variables: $(\hat{M}, \hat{\omega})$

$$\hat{M} := \{(\hat{p}, \hat{q}) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, \hat{p}_1 > \hat{p}_2 > \cdots > \hat{p}_n\}, \quad \hat{\omega} = \sum_{i=1}^n d\hat{q}_i \wedge d\hat{p}_i$$

Formula of action-angle map $R : \hat{M} \to M$

$$q_j = \ln(\sigma_{n+1-j}/\sigma_{n-j}), \quad p_j = \dot{\sigma}_{n+1-j}/\sigma_{n+1-j} - \dot{\sigma}_{n-j}/\sigma_{n-j},$$

$$\sigma_k := \sum_{|I|=k} e^{\sum_{l \in I} \hat{q}_l} \prod_{i \in I, j \notin I} |\hat{p}_i - \hat{p}_j|^{-1} \quad (\forall k = 1, \ldots, n, \quad \sigma_0 := 1)$$

$I \subset \{1, 2, \ldots, n\}$ subset of cardinality $|I| = k$, $\dot{\sigma}_k := \{\sigma_k, \frac{1}{2}\sum_{i=1}^n \hat{p}_i^2\}_{\hat{M}}$

Action-angle map R converts H into free form: $H \circ R = \frac{1}{2}\sum_{i=1}^n \hat{p}_i^2$.

Dual system: $(\hat{M}, \hat{\omega}, \hat{H})$ with $\hat{H} := \sigma_1 = e^{q_n} \circ R = \sum_{i=1}^n e^{\hat{q}_i} \prod_{j \neq i} \frac{1}{|\hat{p}_i - \hat{p}_j|}$

Toda action-angle map and duality from symplectic reduction

Unreduced phase space: $T^*GL(n,\mathbb{R}) \simeq GL(n,\mathbb{R}) \times gl(n,\mathbb{R}) = \{(g,\mathcal{J})\}$ equipped with symplectic form $\Omega := 2d\mathrm{tr}\,(\mathcal{J}g^{-1}dg)$.

Two sets of commuting "free Hamiltonians" $\{\mathcal{H}_k\}$ and $\{\widehat{\mathcal{H}}_k\}$:

$$\mathcal{H}_k(g,\mathcal{J}) := \frac{1}{k}\mathrm{tr}\,(\mathcal{J}^k), \qquad \widehat{\mathcal{H}}_k(g,\mathcal{J}) := m_k((gg^t)^{-1}), \quad k = 1,\dots,n,$$

Notation: $m_k(X) := \det(X_k)$ is k-th leading principal minor of $n \times n$ matrix X.

Reduce by the symmetry group $N_+ \times O(n,\mathbb{R})$. N_+ is upper triangular nilpotent subgroup and (η_+,η_O) from symmetry group acts by the map $\Psi_{(\eta_+,\eta_O)}$:

$$\Psi_{(\eta_+,\eta_O)}(g,\mathcal{J}) := (\eta_+ g\eta_O^{-1}, \eta_O\mathcal{J}\eta_O^{-1}).$$

This Hamiltonian action is generated by the moment map Φ:

$$\Phi(g,\mathcal{J}) = ((g\mathcal{J}g^{-1})_{\text{lower–triangular part}}, -\mathcal{J}_{\text{anti–symmetric part}}).$$

Reduction relevant for Toda is defined by imposing the moment map constraint

$$\Phi(g,\mathcal{J}) = \mu_0 := (I_-, 0), \qquad \text{(Olshanetsky-Perelomov, Adler, Kostant, Symes, ...)}$$

where $I_- := \sum_{i=1}^{n-1} E_{i+1,i}$ contains 1 in its entries just below the diagonal.

Reduced phase space $\Phi^{-1}(\mu_0)/(N_+ \times O(n,\mathbb{R}))$ inherits 2 Abelian Poisson algebras.

First model of the reduced phase space: 'Toda gauge'

By Iwasawa decomposition, any $g \in GL(n, \mathbb{R})$ can be uniquely written as

$$g = g_+ g_A g_O, \qquad (g_+, g_A, g_O) \in N_+ \times A \times O(n, \mathbb{R}).$$

Associate to $(q, p) \in M := \mathbb{R}^n \times \mathbb{R}^n$ the diagonal matrices

$$Q(q) := -\sum_{i=1}^{n} q_{n+1-i} E_{i,i}, \quad P(p) := -\sum_{i=1}^{n} p_{n+1-i} E_{i,i}$$

and define Jacobi matrix (alias Toda Lax matrix, since $H(q,p) = \frac{1}{2}\text{tr}\,(L(q,p)^2)$)

$$L(q, p) := P(p) + e^{-Q(q)/2} I_- e^{Q(q)/2} + e^{Q(q)/2} I_+ e^{-Q(q)/2}.$$

The following manifold S is a global cross section of the orbits of the "gauge group" $N_+ \times O(n, \mathbb{R})$ in the "constraint surface" $\Phi^{-1}(\mu_0)$:

$$S := \{(e^{Q(q)/2}, L(q, p)) \,|\, (q, p) \in M\}.$$

Reduced symplectic form is represented by pull-back $\iota_S^*(\Omega) = \sum_{i=1}^n dp_i \wedge dq_i \equiv \omega$.

The equalities $\quad \iota_S^*(\mathcal{H}_k) = \frac{1}{k}\text{tr}\,(L^k), \quad \iota_S^*(\widehat{\mathcal{H}}_k) = \prod_{j=1}^{k} e^{q_{n+1-j}} \quad$ show that

in terms of model $(S, \iota_S^*(\Omega)) \simeq (M, \omega)$ of reduced phase space, the unreduced free Hamiltonians $\{\mathcal{H}_k\}$ descend to commuting Toda Hamiltonians and $\{\widehat{\mathcal{H}}_k\}$ descend to (functions of) Toda position variables.

All this is well-known. I call S 'Toda gauge': a model of $\Phi^{-1}(\mu_0)/(N_+ \times O(n, \mathbb{R}))$.

Second model of the reduced phase space: ‘Moser gauge’

$\mathbb{R}^n_>$: set of vectors $\hat{p}$ satisfying $\hat{p}_1 > \hat{p}_2 > \cdots > \hat{p}_n$. $\mathbb{R}^n_+$: vectors w having positive components. For $(\hat{p}, w) \in \mathbb{R}^n_> \times \mathbb{R}^n_+$ define $n \times n$ matrices Λ and Γ by

$$\Lambda(\hat{p}) := \mathrm{diag}(\hat{p}_1, \hat{p}_2, \ldots, \hat{p}_n), \qquad \Gamma(\hat{p}, w)_{i,k} := w_i\, (\hat{p}_i)^{k-1} \quad (\texttt{diagonal} \times \texttt{Vandermonde})$$

My main observation: *The manifold*

$$\hat{S} := \{(\Gamma(\hat{p}, w)^{-1}, \Lambda(\hat{p})) \,|\, (\hat{p}, w) \in \mathbb{R}^n_> \times \mathbb{R}^n_+\}$$

is a global cross-section of the orbits of $N_+ \times O(n, \mathbb{R})$ in constraint surface $\Phi^{-1}(\mu_0)$.

The key is to consider Iwasawa decomposition

$$\Gamma(\hat{p}, w)^{-1} = \eta_+(\hat{p}, w)\rho(\hat{p}, w)\eta_O(\hat{p}, w) \text{ with } \rho(\hat{p}, w) = \mathrm{diag}(\rho_1(\hat{p}, w), \ldots, \rho_n(\hat{p}, w)).$$

Fact: $\eta_O(\hat{p}, w)\Lambda(\hat{p})\eta_O(\hat{p}, w)^{-1}$ is Jacobi matrix, determines $(\hat{p}, w)$ up to scale of w.

Then unique gauge transformation from $\hat{S}$ to S yields a map

$$\mathcal{R} : \hat{S} \to S, \qquad (\hat{p}, w) \mapsto (e^{Q(q)/2}, L(q, p)) = (\rho(\hat{p}, w), \eta_O(\hat{p}, w)\Lambda(\hat{p})\eta_O(\hat{p}, w)^{-1}).$$

It is EASY to find this map explicitly since $\Gamma(\hat{p}, w)$ is $\texttt{diagonal} \times \texttt{Vandermonde}$.

Using Cauchy-Binet, trivial calculation of $\hat{\mathcal{H}}_k(g, \mathcal{J}) = m_k((gg^t)^{-1})$ in the two gauges gives

$$\prod_{j=1}^{k} e^{q_{n+1-j}} \circ \mathcal{R} = m_k(\Gamma(\hat{p}, w)^t \Gamma(\hat{p}, w)) = \sum_{|I|=k} \Big(\prod_{l \in I} w_l^2 \prod_{\substack{i,j \in I \\ i \neq j}} |\hat{p}_i - \hat{p}_j| \Big).$$

To finish, parametrize Moser’s variables $(\hat{p}, w)$ by Darboux coordinates $(\hat{p}, \hat{q})$.

Ruijsenaars' action-angle map and duality from reduction

Reduced symplectic form is easily calculated in the Moser gauge

$$\iota_{\widehat{S}}^*(\Omega) = 2\sum_{i=1}^n d\ln w_i \wedge d\widehat{p}_i + \sum_{\substack{j,k=1\\ j\neq k}}^n \frac{d\widehat{p}_j \wedge d\widehat{p}_k}{\widehat{p}_j - \widehat{p}_k} \qquad \texttt{(thanks to C. Klimcik)}$$

Corresponding Poisson brackets: $\{\widehat{p}_i, \widehat{p}_j\} = 0$, $\{\widehat{p}_i, w_j\} = \frac{w_j}{2}\delta_{ij}$, $\{w_j, w_k\} = \frac{1}{2}\frac{w_j w_k}{\widehat{p}_j - \widehat{p}_k}$.
These variables linearize the Toda flows, whose Hamiltonians become on $\widehat{S}$

$$\iota_{\widehat{S}}^*(\mathcal{H}_k) = \frac{1}{2}\sum_{i=1}^n (\widehat{p}_i)^k.$$

Toda action-angle variables $(\widehat{p}, \widehat{q})$ are obtained by the parametrization

$$w_i(\widehat{p}, \widehat{q}) := e^{\frac{1}{2}\widehat{q}_i} \prod_{\substack{j=1\\ j\neq i}}^n |\widehat{p}_i - \widehat{p}_j|^{-\frac{1}{2}}, \qquad (\widehat{p}, \widehat{q}) \in \mathbb{R}^n_> \times \mathbb{R}^n \equiv \widehat{M},$$

which brings $\iota_{\widehat{S}}^*(\Omega)$ into Darboux form $\widehat{\omega} = \sum_{i=1}^n d\widehat{q}_i \wedge d\widehat{p}_i$.

Map $\mathcal{R} : \widehat{S} \to S$ is automatically symplectomorphism, and "explains" Ruijsenaars' formula. Reduced Hamiltonians $\iota_{\widehat{S}}^*(\widehat{\mathcal{H}}_k)$ are Ruijsenaars' dual Hamiltonians.
Toda position variables q_k are action variables of main dual Hamiltonian:

$$\hat{H} = e^{q_n} \circ \mathcal{R} = \iota_{\widehat{S}}^*(\widehat{\mathcal{H}}_1) = \sum_{i=1}^n w_i^2 = \sum_{i=1}^n e^{\widehat{q}_i} \prod_{j\neq i} \frac{1}{|\widehat{p}_i - \widehat{p}_j|}.$$

CONCLUDING REMARKS

The same general ideas have been applied to explain almost all the other known duality relations, and were also used to find new ones. For example, one of the most complicated cases is the relation between the trigonometric Ruijsenaars-Schneider system

$$H_{\text{trigo}-\text{RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j \neq k} \left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_j)} \right]^{\frac{1}{2}}$$

and the **physically very different** dual system

$$\widehat{H}_{\text{trigo}-\text{RS}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j \neq k} \left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_j)} \right]^{\frac{1}{2}}$$

This was enlightened by reduction of Heisenberg double of P-L $U(n)$

There are still many open problems, including the self-duality of

$$H_{\text{hyp}-\text{RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j \neq k} \left[1 + \frac{\sinh^2 x}{\sinh^2(q_k - q_j)} \right]^{\frac{1}{2}}$$